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Bornological convergences

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Abstract

We study a family of convergences (actually pretopologies) in the hyperspace of a metric space that are generated by covers of the space. This family includes the Attouch–Wets, Fell, and Hausdorff metric topologies as well as the lower Vietoris topology. The unified approach leads to new developments and puts into perspective some classical results.

Keywords: Convergences; Hyperspaces; Covers; Bornologies; Pretopology; Hausdorff metric topology; Uniform structure; Totally bounded sets

0. Introduction

Let (X, d) be a metric space. For subsets C and D of X, the Hausdorff distance between C and D is given by $h(C, D) = \inf\{\varepsilon > 0\colon C \subseteq B(D, \varepsilon) \text{ and } D \subseteq B(C, \varepsilon)\}$, where $B(A, \varepsilon)$ is the ε-enlargement of the set A of radius ε. The Hausdorff distance induces a convergence H on the power set 2^X by defining $A_t \xrightarrow{\mathrm{H}} A$ whenever $h(A_t, A) \to 0$. However, this convergence works well only when restricted to bounded (closed) subsets. For

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unbounded sets convergence in the Hausdorff distance turns out to be too strong. There are simple examples of sequences of subsets (such as the lines $y = x/n$ in the plane) that do not converge with respect to the Hausdorff distance, but should "reasonably" converge.

One solution to overcome this difficulty is to modify the convergence H by elements of a properly chosen family $\mathcal{S} \subseteq 2^X$. The most celebrated example of this approach is the so-called Attouch–Wets convergence AW, also called bounded-Hausdorff convergence. This convergence was initially introduced by Mosco [19] and studied later by Attouch and Wets (see [1–3]). Fixing $x_0 \in X$, we say that a net (A_t) of subsets AW-converges to $A \subseteq X$ if for every $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$A_t \cap B(x_0, n) \subseteq B(A, \varepsilon) \quad \text{and} \quad A \cap B(x_0, n) \subseteq B(A_t, \varepsilon), \quad \text{eventually.}$$

In this case convergence in the Hausdorff distance has been modified by the family $\mathcal{S}$ of bounded sets.

Attouch–Wets convergence has been intensively investigated since the mid 80's (see [5] for references) and applied to study approximation and optimization problems. It turned out that besides the family of bounded subsets, other families were also useful for modifying the convergence H. For instance, in the case of a Banach space X one can consider the family $\mathcal{S}$ of all norm (or weakly) compact subsets of X [7].

In this paper we present a general theory of $\tilde{\mathcal{S}}$-convergences which are modified H-convergences through the use of various families $\mathcal{S}$ of subsets of X. It turns out that all such convergences are pretopologies and that the families $\mathcal{S}$ are essentially bornologies on X. This explains the title of the paper.

Besides the trivial bornology ($\mathcal{S} = 2^X$) and the bornology of bounded sets, the bornologies of finite sets, compact sets and totally bounded sets are also of general interest: their common lower part is the lower Vietoris topology (Proposition 2.2.) while the corresponding upper parts are, respectively, the cofinite topology, the co-compact topology (so that the Fell topology is a particular $\tilde{\mathcal{S}}$-convergence) and a newly identified object in the case of totally bounded sets.

The paper is organized as follows: in Section 1 we recall some definitions and fix the notation. Section 2 is devoted to lower bornological convergences, called $\mathcal{S}^-$-convergences. We study relationships between $\mathcal{S}^-$-convergences and other "lower" convergences, such as the lower Vietoris topology. Moreover, topological and uniform properties of $\mathcal{S}^-$ are investigated. In Section 3 we discuss upper bornological convergences $\mathcal{S}^+$. We show that there are distinct differences in the behaviour of $\mathcal{S}^+$ as compared to $\mathcal{S}^-$. And finally, in Section 4 we consider the convergence $\tilde{\mathcal{S}}$ as the supremum of $\mathcal{S}^-$ and $\mathcal{S}^+$.

1. Preliminaries

Let Z be a nonempty set and let φZ denote the family of all filters on Z. For each $z \in Z$, let $\mathcal{U}_l(z)$ denote the ultrafilter generated by $\{z\}$. A *convergence* on Z is a mapping π from φZ to the family 2^Z of all subsets of Z which satisfies the following conditions:

(i) $z \in \pi(\mathcal{U}_l(z))$ for all z in Z;
(ii) $\mathcal{F} \subseteq \mathcal{G}$ implies $\pi(\mathcal{F}) \subseteq \pi(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \varphi Z$.

The pair (Z,π) is called a *convergence space*. If $\mathcal{F} \in \varphi Z$ and $z \in \pi(\mathcal{F})$, then we say that $\mathcal{F}$ π*-converges* to z and we write $z \in \pi\text{-}\lim \mathcal{F}$. The notion of convergence can be equivalently formulated in terms of nets (see, e.g., [12,13]). Thus every convergence π on Z can be treated as a mapping from the family of nets on Z into 2^Z.

In this paper we will use both the net and the filter terminology.

If π_1 and π_2 are two convergences on Z, we say that π_2 is *finer* than π_1, or that π_1 is *coarser* than π_2, and write $\pi_1 \leqslant \pi_2$, provided $z \in \pi_2\text{-}\lim z_t \Rightarrow z \in \pi_1\text{-}\lim z_t$ for every net (z_t) on Z. Let π_i, $i \in I$, be a family of convergences on Z. For each filter $\mathcal{F}$ on Z we denote $\pi(\mathcal{F}) = \bigcap_{i\in I} \pi_i(\mathcal{F})$. Then π is a convergence on Z finer than each convergence π_i, $i \in I$. We denote this uniquely defined convergence with $\bigvee_{i\in I} \pi_i$ and call it the *supremum of the convergences* π_i, $i \in I$.

Let (Z,π) be a convergence space and $z \in Z$. Let $\mathcal{N}_\pi(z)$ be the filter obtained by intersecting all filters that π-converge to z. This filter is called the π*-neighbourhood filter* at z, and its elements are π*-neighbourhoods* of z.

A convergence π on Z is called *pretopological* (or a *pretopology*) if $\mathcal{N}_\pi(z)$ π-converges to z for each $z \in Z$. If π is a pretopology on Z, the pair (Z,π) is called a *pretopological space*.

Let $(X,\mathfrak{U})$ be a quasi-uniform space and $\mathfrak{U}^{-1} = \{U^{-1}\colon U \in \mathfrak{U}\}$ be the conjugate quasi-uniformity, where $U^{-1} = \{(y,x)\colon (x,y) \in U\}$. If $\mathcal{S}$ is a nonempty family of subsets of X, we may consider the following three convergences on 2^X. Let (A_t) be a net of subsets of X and $A \subseteq X$. We say that the net (A_t):

- $\mathcal{S}^-$*-converges* to A, and we write $A \in \mathcal{S}^-\text{-}\lim A_t$ or $A_t \overset{\mathcal{S}^-}{\to} A$, provided for each $S \in \mathcal{S}$ and $U \in \mathfrak{U}$ there exists t_0 such that $A \cap S \subseteq U^{-1}(A_t)$ for every $t \geqslant t_0$;
- $\mathcal{S}^+$*-converges* to A, and we write $A \in \mathcal{S}^+\text{-}\lim A_t$ or $A_t \overset{\mathcal{S}^+}{\to} A$, provided for each $S \in \mathcal{S}$ and $U \in \mathfrak{U}$ there exists t_0 such that $A_t \cap S \subseteq U(A)$ for every $t \geqslant t_0$;
- $\tilde{\mathcal{S}}$*-converges* to A provided it $\mathcal{S}^-$-converges to A and $\mathcal{S}^+$-converges to A,

where $U(A) = \{y \in X\colon (x,y) \in U \text{ for some } x \in A\}$, $U^{-1}(A) = \{y \in X\colon (x,y) \in U^{-1} \text{ for some } x \in A\}$.

For similar ideas see [4,6,7,20,22].

In this paper the underlying space X is a metric space. We use the natural uniformity of X, and the definitions of $\mathcal{S}^-$, $\mathcal{S}^+$ and $\tilde{\mathcal{S}}$ can be reformulated as follows:

$$A_t \overset{\mathcal{S}^-}{\to} A \quad\Leftrightarrow\quad \forall S \in \mathcal{S},\ \forall \varepsilon > 0,\ \exists t_0,\ \forall t \geqslant t_0,\ A \cap S \subseteq A_t^\varepsilon,$$

$$A_t \overset{\mathcal{S}^+}{\to} A \quad\Leftrightarrow\quad \forall S \in \mathcal{S},\ \forall \varepsilon > 0,\ \exists t_0,\ \forall t \geqslant t_0,\ A_t \cap S \subseteq A^\varepsilon,$$

and

$$A_t \overset{\tilde{\mathcal{S}}}{\to} A \quad\Leftrightarrow\quad \forall S \in \mathcal{S},\ \forall \varepsilon > 0,\ \exists t_0,\ \forall t \geqslant t_0,\ A \cap S \subseteq A_t^\varepsilon \text{ and } A_t \cap S \subseteq A^\varepsilon,$$

where $A^\varepsilon = B(A,\varepsilon) = \bigcup_{x\in A} B(x,\varepsilon)$ and $B(x,\varepsilon)$ is the open ball with center x and radius ε.

Note that if X is a metric space and $\mathcal{S} = \{X\}$ then the $\tilde{\mathcal{S}}$-convergence on the space of closed bounded subsets of X is simply the H-convergence, i.e., the convergence in the

Hausdorff metric. We also denote $\mathrm{H}^- = \{X\}^-$, $\mathrm{H}^+ = \{X\}^+$, and V^-, the lower Hausdorff, the upper Hausdorff, and the lower Vietoris convergence, respectively.

We now define some useful set-theoretical operations on families of subsets:

$$\downarrow\mathcal{S} = \{A \subseteq X\colon\ A \subseteq S \text{ for some } S \in \mathcal{S}\},$$
$$\Sigma(\mathcal{S}) = \{S_1 \cup S_2 \cup \cdots \cup S_n\colon\ S_i \in \mathcal{S} \text{ for } i = 1, 2, \ldots, n;\ n \in \mathbb{N}\},$$

and we have the following properties:

$$\mathcal{S} \subseteq \downarrow\mathcal{S}, \quad \mathcal{S} \subseteq \Sigma(\mathcal{S}), \quad \downarrow\Sigma(\mathcal{S}) = \Sigma(\downarrow\mathcal{S}),$$
$$\downarrow\{X\} = 2^X, \quad \downarrow\{\emptyset\} = \{\emptyset\}.$$

By $\mathrm{s}(X)$, $\mathrm{cl}(X)$, $\mathrm{b}(X)$, $\mathrm{tb}(X)$ and $\mathrm{c}(X)$ we denote the families of all singletons of X, closed subsets of X, bounded subsets of X, totally bounded subsets of X and compact subsets of X, respectively.

Let $\mathcal{S}, \mathcal{W} \subseteq 2^X$ be nonempty. We say that $\mathcal{S}$ *refines* $\mathcal{W}$, and write $\mathcal{S} \leqslant \mathcal{W}$, if for every $S \in \mathcal{S}$ there exists $W \in \mathcal{W}$ with $S \subseteq W$. Observe that $\mathcal{S} \leqslant \mathcal{W}$ if and only if $\mathcal{S} \subseteq \downarrow\mathcal{W}$, and as a consequence: $\mathcal{S} \leqslant \mathcal{W}$ and $\mathcal{W} \leqslant \mathcal{S}$ if and only if $\downarrow\mathcal{S} = \downarrow\mathcal{W}$.

A *bornology* on a set X is a family $\mathcal{S}$ of subsets of X such that:

(1) $\mathcal{S}$ is a cover of X,
(2) $\mathcal{S}$ is closed under subsets and
(3) $\mathcal{S}$ is closed under finite unions (see [15]).

The set convergences defined above are generated by families of subsets which are essentially bornologies.

From now on we assume that X is a metric space and d its metric.

2. $\mathcal{S}^-$-convergences

We begin with the following observations on $\mathcal{S}^-$-convergences:

$$\emptyset \in \mathcal{S}^-\text{-}\lim A_t \quad \text{for every } \mathcal{S} \text{ and every net } (A_t), \tag{2.1}$$
$$\text{if } A \in \mathcal{S}^-\text{-}\lim A_t \text{ and } B \subseteq A, \quad \text{then } B \in \mathcal{S}^-\text{-}\lim A_t, \tag{2.2}$$
$$\mathcal{S}^- = (\downarrow\mathcal{S})^- = \big(\Sigma(\mathcal{S})\big)^- = \big(\Sigma(\downarrow\mathcal{S})\big)^- = \big(\downarrow\big(\Sigma(\mathcal{S})\big)\big)^-, \tag{2.3}$$
$$\text{if } \mathcal{S} \leqslant \mathcal{W}, \quad \text{then } \mathcal{S}^- \leqslant \mathcal{W}^-. \tag{2.4}$$

Proposition 2.1. *Let $\mathcal{S}$ be a cover of X. Then*

(i) $\mathrm{V}^- \leqslant \mathcal{S}^- \leqslant \mathrm{H}^-$;
(ii) $\mathrm{V}^- = \mathrm{s}(X)^-$ *and* $\mathrm{H}^- = \{X\}^-$.

Proof. Let A_t and A be elements of 2^X and $\varepsilon > 0$.

(i) If $A_t \xrightarrow{\mathrm{H}^-} A$, it is clear that $A_t \xrightarrow{\mathcal{S}^-} A$, as for every $S \in \mathcal{S}$, $A \cap S \subseteq A \subseteq A_t^\varepsilon$ eventually. Suppose that $A_t \xrightarrow{\mathcal{S}^-} A$ and that $B(x,\varepsilon) \cap A \neq \emptyset$; pick $y \in B(x,\varepsilon) \cap A$, $S \in \mathcal{S}$ with $y \in S$ and choose $r > 0$ such that $d(x,y) + r < \varepsilon$; then there exists t_0 such that $A \cap S \subseteq A_t^r$ for all $t \geqslant t_0$ and for every such t we can choose $a_t \in A_t$ with $d(y,a_t) < r$. Thus $d(x,a_t) \leqslant d(x,y) + d(y,a_t) < d(x,y) + r < \varepsilon$ and $B(x,\varepsilon) \cap A_t \neq \emptyset$. This shows that $A_t \xrightarrow{\mathrm{V}^-} A$.

(ii) By (i) we only need to prove that $\mathrm{s}(X)^- \leqslant \mathrm{V}^-$. Let $A_t \xrightarrow{\mathrm{V}^-} A$ and pick $x_0 \in X$ and $\varepsilon > 0$. We can suppose that $x_0 \in A$; then $B(x_0,\varepsilon) \cap A \neq \emptyset$ and there exists t_0 such that $B(x_0,\varepsilon) \cap A_t \neq \emptyset$ for all $t \geqslant t_0$. This shows that $x_0 \in A_t^\varepsilon$ for all $t \geqslant t_0$. Also, it is obvious that $\mathrm{H}^- = \{X\}^-$. $\square$

Remark. If $\mathcal{S}$ contains a nonempty set then the $\mathcal{S}^-$-convergence is admissible if and only if $\mathcal{S}$ is a cover of X. Recall that a convergence on a hyperspace is called *admissible* if the mapping $x \mapsto \{x\}$ is an embedding. It is not hard to show that the mapping $x \mapsto \{x\}$ is $\mathcal{S}^-$-continuous for every family $\mathcal{S}$ and if $\mathcal{S}$ is a cover of X then the inverse mapping $\{x\} \mapsto x$ is continuous as well. Consequently, $\mathcal{S}^-$ is admissible provided $\mathcal{S}$ is a cover of X. Observe that this fact follows also from Proposition 2.1(i), because V^- and H^- are admissible. Now assume that for $\mathcal{S} \neq \{\emptyset\}$, $\mathcal{S}^-$ is admissible and suppose that $\mathcal{S}$ is not a cover of X, i.e., there exists $x \in X \setminus \bigcup \mathcal{S}$. Take any y from $\bigcup \mathcal{S}$ and put $x_n = y$ for $n \in \mathbb{N}$. Then $\{x_n\} \xrightarrow{\mathcal{S}^-} \{x\}$ but $x_n \not\to x$, a contradiction.

Proposition 2.2. $\mathrm{V}^- = \mathcal{S}^-$ *if and only if every* $S \in \mathcal{S}$ *is totally bounded.*

Proof. Suppose each $S \in \mathcal{S}$ is totally bounded and let $A_t \xrightarrow{\mathrm{V}^-} A$, $S \in \mathcal{S}$ and $\varepsilon > 0$; then $A \cap S$ is totally bounded and there exist $x_1, \ldots, x_n \in A$ such that $A \cap S \subseteq \bigcup_1^n B(x_i, \varepsilon/2)$. By V^--convergence of (A_t) to A,

$$\exists t_0,\ \forall t \geqslant t_0,\ A_t \cap B(x_i, \varepsilon/2) \neq \emptyset \quad \text{for } i = 1, \ldots, n;$$

pick $y_i \in A_t \cap B(x_i, \varepsilon/2)$; then

$$A \cap S \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon/2) \subseteq \bigcup_{i=1}^n B(y_i, \varepsilon) \subseteq A_t^\varepsilon \quad \text{for all } t \geqslant t_0.$$

Conversely, suppose that there exists $S \in \mathcal{S}$ which is not totally bounded; then there exists $\varepsilon > 0$ such that $S \nsubseteq \bigcup_1^n B(x_i, \varepsilon)$ for all $\{x_1, \ldots, x_n\} \subseteq S$. If $t = \{x_1, \ldots, x_n\}$ and $s = \{y_1, \ldots, y_m\}$ belong to $\bigcup_1^\infty X^k$, put $t \leqslant s$ if and only if $t \subseteq s$ and define the net (A_t) on 2^X by

$$A_t = \bigcup_{i=1}^n B(x_i, \varepsilon/2).$$

Then (A_t) is not $\mathcal{S}^-$-convergent to S, as S is not contained in any $A_t^{\varepsilon/2}$, while $A_t \xrightarrow{\mathrm{V}^-} S$; to see this pick $y \in S$, $\sigma > 0$ and put $t_0 = \{y\}$. Then $A_t \cap B(y, \sigma) \neq \emptyset$ for all $t \geqslant t_0$. $\square$

Corollary 2.3. *For every cover $\mathcal{S}$*:

(i) (*localization*) *If $A \subseteq X$ is such that $A \cap S$ is totally bounded for each $S \in \mathcal{S}$, then $A_t \xrightarrow{V^-} A$ if and only if $A_t \xrightarrow{\mathcal{S}^-} A$*;

(ii) *If A is totally bounded, then $A_t \xrightarrow{V^-} A$ if and only if $A_t \xrightarrow{\mathcal{S}^-} A$.*

Now, given two covers $\mathcal{S}$ and $\mathcal{W}$, what are the conditions for $\mathcal{S}$ and $\mathcal{W}$ to generate the same convergence?

We know that if $\mathcal{S}$ refines $\mathcal{W}$, then $\mathcal{S}^- \leqslant \mathcal{W}^-$ but the reverse implication is not true in general: take $X = \mathbb{R}^2$, $\mathcal{W}$ the cover of all singletons and $\mathcal{S} = \mathcal{W} \cup \{\text{the unit disc}\}$. By Proposition 2.2, $\mathcal{S}^- = \mathcal{W}^- = V^-$, while $\mathcal{S}$ (or $\Sigma(\mathcal{S})$) is not a refinement of $\mathcal{W}$.

We can, however, proceed in the spirit of Proposition 2.2 provided the right generalization of total boundedness is introduced:

Definition 2.4. Let $\mathcal{S}$ be a cover of X; a subset A of X is totally bounded with respect to $\mathcal{S}$, or A is *$\mathcal{S}$-totally bounded*, if

$$\forall \varepsilon > 0,\ \exists S_1, \ldots, S_k \in \mathcal{S},\ A \subseteq (A \cap S_1)^{\varepsilon} \cup \cdots \cup (A \cap S_k)^{\varepsilon}.$$

We note the following properties of $\mathcal{S}$-total boundedness:

(a) *Every subset of each $S \in \mathcal{S}$ is $\mathcal{S}$-totally bounded*;
(b) *If A_1 and A_2 are $\mathcal{S}$-totally bounded, then $A_1 \cup A_2$ is $\mathcal{S}$-totally bounded*;
(c) *If $\mathcal{S}$ is the cover of all singletons, $\mathcal{S}$-total boundedness is the usual total boundedness*;
(d) *If the cover $\mathcal{S}$ refines the cover $\mathcal{W}$, then $\mathcal{S}$-totally bounded sets are also $\mathcal{W}$-totally bounded.*

Theorem 2.5. *Let $\mathcal{S}$ and $\mathcal{W}$ be covers of X. The following are equivalent*:

(i) *For all $A \subseteq X$ and all $S \in \mathcal{S}$, $A \cap S$ is $\mathcal{W}$-totally bounded*;
(ii) *The family of $\mathcal{S}$-totally bounded subsets of X is included in the family of $\mathcal{W}$-totally bounded subsets of X*;
(iii) *$\mathcal{S}^- \leqslant \mathcal{W}^-$.*

Proof. (i) $\Rightarrow$ (ii) Let $B \subset X$ be $\mathcal{S}$-totally bounded and pick $\varepsilon > 0$; there exist $S_1, \ldots, S_m \in \mathcal{S}$ such that $B \subset (B \cap S_1)^{\varepsilon/2} \cup \cdots \cup (B \cap S_m)^{\varepsilon/2}$; by (i) the sets $B \cap S_1, \ldots, B \cap S_m$ are $\mathcal{W}$-totally bounded and there exist $W_1^1, \ldots, W_{k_1}^1, \ldots, W_1^m, \ldots, W_{k_m}^m \in \mathcal{W}$ such that for $j = 1, 2, \ldots, m$,

$$B \cap S_j \subseteq \big(B \cap S_j \cap W_1^j\big)^{\varepsilon/2} \cup \cdots \cup \big(B \cap S_j \cap W_{k_j}^j\big)^{\varepsilon/2};$$

thus

$$B \subseteq \bigcup_{j=1}^{m} \bigcup_{r=1}^{k_j} \big(B \cap S_j \cap W_r^j\big)^{\varepsilon} \subseteq \bigcup_{j=1}^{m} \bigcup_{r=1}^{k_j} \big(B \cap W_r^j\big)^{\varepsilon}.$$

(ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii) Suppose $A_t \xrightarrow{\mathcal{W}^-} A$ and let $S \in \mathcal{S}$ and $\varepsilon > 0$ be given. There exist $W_1, \ldots, W_k \in \mathcal{W}$ such that $A \cap S \subseteq \bigcup_1^k (A \cap S \cap W_j)^{\varepsilon/2}$ and t_0 such that $A \cap W_j \subseteq A_t^{\varepsilon/2}$ for all $t \geqslant t_0$ and $j = 1, \ldots, m$; therefore

$$A \cap S \subseteq \bigcup_{j=1}^{k} (A \cap S \cap W_j)^{\varepsilon/2} \subseteq \bigcup_{j=1}^{k} (A \cap W_j)^{\varepsilon/2} \subseteq A_t^{\varepsilon}$$

for all $t \geqslant t_0$. Hence $A_t \xrightarrow{\mathcal{S}^-} A$.

(iii) $\Rightarrow$ (i) Suppose there exists $A \subseteq X$ such that $T = A \cap S$ is not $\mathcal{W}$-totally bounded for some $S \in \mathcal{S}$; then there exists $\varepsilon > 0$ such that $T \nsubseteq \bigcup_1^m (T \cap W_i)^{\varepsilon}$ for all $W_1, \ldots, W_m \in \mathcal{W}$; put $t = \{W_1, \ldots, W_m\} \subset \mathcal{W}$ and order the set $\bigcup_1^{\infty} \mathcal{W}^n$ as in the proof of Proposition 2.2. Put $B_t = (T \cap W_1) \cup \cdots \cup (T \cap W_m)$; then it is clear that $B_t \xrightarrow{\mathcal{W}^-} T$ and that $B_t \overset{\mathcal{S}^-}{\nrightarrow} T$, as $T \cap S = T \nsubseteq B_t^{\varepsilon}$. □

Let us denote by $\mathcal{S}_\star$ the family of all $\mathcal{S}$-totally bounded subsets of X.

Corollary 2.6.

(i) $\mathcal{S}^- = \mathcal{W}^-$ *if and only if* $\mathcal{S}_\star = \mathcal{W}_\star$;
(ii) *There are* $\mathcal{S}^-$*-convergences that are not comparable*: *take, for example,* $X = \mathbb{R}^2$ *and* $\mathcal{S}$ *and* $\mathcal{W}$ *the covers consisting of horizontal and vertical lines, respectively*;
(iii) *Given* $A \subseteq X$, *every subset of* A *is* $\mathcal{S}$*-totally bounded if and only if* $\mathcal{S}^- = (\mathcal{S} \cup \{A\})^-$;
(iv) $\mathcal{S}^- = \mathrm{H}^-$ *if and only if every subset of* X *is* $\mathcal{S}$*-totally bounded, as* $\{X\}_\star = 2^X$;
(v) (*localization*) *If* $A \subset X$ *is such that for every* $W \in \mathcal{W}$ *the set* $A \cap W$ *is* $\mathcal{S}$*-totally bounded, then* $A_t \xrightarrow{\mathcal{S}^-} A \Rightarrow A_t \xrightarrow{\mathcal{W}^-} A$.

In the list, (a)–(d), of properties of $\mathcal{S}$-total boundedness, we did not mention subsets; it turns out that, in general, $\mathcal{S}_\star$ is not closed under subsets (see Example 2.7 below).

Another natural generalization of total boundedness is the following: say that a subset B of X is *weakly* $\mathcal{S}$*-totally bounded* if $\forall \varepsilon > 0$, $\exists S_1, \ldots, S_k \in \mathcal{S}$ such that $B \subseteq S_1^{\varepsilon} \cup \cdots \cup S_k^{\varepsilon}$. It is clear that if $\mathcal{S}$ is the cover of all singletons, weak $\mathcal{S}$-total boundedness and $\mathcal{S}$-total boundedness agree. Denote by $\mathcal{S}^\star$ the collection of all weakly $\mathcal{S}$-totally bounded subsets of X. Then $\mathcal{S}^\star$ is closed under finite unions and subsets, $\mathcal{S}_\star \subseteq \mathcal{S}^\star$ and $(\mathcal{S}^\star)^\star = \mathcal{S}^\star$. The next example shows however, that $\mathcal{S}^\star$ is too large for our purposes.

Example 2.7. Let $X = \mathbb{R}^2$, $\mathcal{S}$ the cover of vertical lines. The stripe $B = \{(x, y)\colon\ 0 \leqslant x \leqslant 1\}$ is $\mathcal{S}$-totally bounded, but its subset $A = \{(x, y) \in B\colon\ 0 < x \leqslant 1,\ \ y = 1/x\}$ is not; by Corollary 2.6, $(\mathcal{S} \cup \{B\})^-$-convergence is strictly stronger than $\mathcal{S}^-$-convergence, while $\mathcal{S}^\star = (\mathcal{S} \cup \{B\})^\star$, as B is weakly $\mathcal{S}$-totally bounded.

As an application of Theorem 2.5, we have

Proposition 2.8. *The following are equivalent*:

(i) $\mathcal{S}_\star$ *is closed under subsets*;
(ii) $\mathcal{S}^- = \mathcal{S}_\star^-$;
(iii) $\mathcal{S}_\star = (\mathcal{S}_\star)_\star$.

We now turn our attention to the question of convergence and uniform properties of $\mathcal{S}^-$. The first observation is that $\mathcal{S}^-$-convergence need not verify the iterated limit condition. Consequently, it is not topological in general.

Example 2.9. Let $X = \mathbb{R}^2$, $\mathcal{S}$ the cover of horizontal lines and put $A = \{(x, y)\colon\ x \leqslant 0, y = 0\}$; for each $n \in \mathbb{N}$ consider

$$B_n = \left\{(x, y)\colon\ x \leqslant 0,\ y = \frac{1}{2ne^x}\right\}.$$

Then $B_n \xrightarrow{\mathcal{S}^-} A$, as for each horizontal line l and $\varepsilon > 0$, either $A \cap l = \emptyset$ or $A \cap l = A \subseteq B_n^\varepsilon$ for all $n > 1/(2\varepsilon)$.

Now, for each n and m in $\mathbb{N}$, consider the m horizontal lines

$$k_i = \left\{(x, y)\colon\ y = \frac{i}{2nm}\right\} \quad (1 \leqslant i \leqslant m)$$

and put $C_m^n = B_n \cap \{k_1 \cup \cdots \cup k_m\}$. Then $C_m^n \xrightarrow{\mathcal{S}^-} B_n$ as $m \to \infty$. Also the C_m^n's do not $\mathcal{S}^-$-converge to A as A is unbounded and each C_m^n is finite.

However, nontopological convergences are not unusual in the theory of hyperspaces. For instance, the well-known Kuratowski convergence is also not topological in general.

Nontopological convergences build a wide spectrum: from very general (no constraints at all) to fairly specialized (such as pretopologies) which are close to ordinary topologies (see, e.g., [10] for more details on general convergences).

By definition, $\mathcal{S}^-$-convergence is a modification of the lower Hausdorff convergence H^- by the family $\mathcal{S}$. It is well known (see, e.g., [11]) that the convergence H^- can be described in terms of quasi-uniformities. The family $\{\mathbf{H}_\varepsilon^-\colon\ \varepsilon > 0\}$, where $\mathbf{H}_\varepsilon^- = \{(A, B) \in 2^X \times 2^X\colon\ A \subseteq B^\varepsilon\}$, is a base of a quasi-uniformity on 2^X compatible with H^-.

Let us consider the family $\mathbb{S}^- = \{\mathbf{S}^-(S_1, \ldots, S_n; \varepsilon)\colon\ S_1, \ldots, S_n \in \mathcal{S}$ and $\varepsilon > 0\}$, where

$$\mathbf{S}^-(S_1, \ldots, S_n; \varepsilon) = \left\{(A, B) \in 2^X \times 2^X\colon\ A \cap S_i \subseteq B^\varepsilon \text{ for } i = 1, \ldots, n\right\}.$$

One might presume that this family generates a kind of uniform structure on 2^X that is related to $\mathcal{S}^-$. However, since $\mathcal{S}^-$ is not topological in general, this structure cannot be expected to be quasi-uniform (quasi-uniformities are always compatible with topological convergences).

Observe that the family $\mathbb{S}^-$ is a filter-base. Let $\mathbf{S}^-$ denote the filter on $2^X \times 2^X$ generated by $\mathbb{S}^-$. Of course, each element of $\mathbf{S}^-$ contains the diagonal of $2^X \times 2^X$. Consequently, the family $\mathbf{S}^-$ is a pretopological uniform structure on 2^X. It can be shown that in general the filter $\mathcal{S}^-$ is neither symmetric (i.e., $\mathcal{S}^- \neq (\mathcal{S}^-)^{-1}$) nor composable ($\mathcal{S}^- \nsubseteq \mathcal{S}^- \circ \mathcal{S}^-$). This means that $\mathcal{S}^-$ is neither a uniformity nor a quasi-uniformity in general.

Now we come to the question of how $\mathcal{S}^-$ is related to $\mathcal{S}^-$-convergence. It is known (see, e.g., [16,18]) that every pretopological uniform structure induces a pretopology on the underlying space. Thus if $\mathcal{S}^-$ is compatible with $\mathcal{S}^-$, the convergence $\mathcal{S}^-$ is a pretopology on 2^X.

Pretopologies are convergences which have convergent neighbourhood filters (see [10]). Thus they can be described in terms of total systems of neighbourhoods. However, the main difference with topologies is that pretopologies need not admit neighbourhood filters generated by open sets (this property is equivalent to the iterated limit condition).

The total system of neighbourhoods of the pretopology $\lambda(\mathcal{S}^-)$ induced by $\mathcal{S}^-$ on 2^X is equal to $\{\mathcal{S}^-(A)\colon A \subseteq X\}$, where $\mathcal{S}^-(A)$ denotes the section filter of $\mathcal{S}^-$ at A, i.e., the filter on 2^X generated by the sets

(N) $\mathbf{S}^-(S_1, \ldots, S_n; \varepsilon)(A) = \{B \subset X\colon A \cap S_i \subseteq B^\varepsilon \text{ for } i = 1, \ldots, n\}$,

where $S_1, \ldots, S_n \in \mathcal{S}$ and $\varepsilon > 0$.

Lemma 2.10. *The pretopological uniform structure $\mathcal{S}^-$ is compatible with the convergence $\mathcal{S}^-$, i.e., $\lambda(\mathcal{S}^-) = \mathcal{S}^-$.*

Proof. Let $S_1, \ldots, S_n \in \mathcal{S}$ and $\varepsilon > 0$ be given. Take a net (A_t) of subsets of X. It follows immediately from (N) that $A_t \in \mathbf{S}^-(S_1, \ldots, S_n; \varepsilon)(A)$ if and only if $A \cap S_i \subseteq A_t^\varepsilon$ for $i = 1, \ldots, n$. Consequently, $\lambda(\mathcal{S}^-) = \mathcal{S}^-$. □

From Lemma 2.10 we infer the following

Theorem 2.11. *The convergence $\mathcal{S}^-$ is a pretopology on 2^X. For each $A \subseteq X$ the family*

$$\mathcal{B}_{\mathcal{S}^-}(A) = \left\{\mathbf{S}^-(S_1, \ldots, S_n; \varepsilon)(A)\colon S_1, \ldots, S_n \in \mathcal{S} \text{ and } \varepsilon > 0\right\}$$

is a local base of $\mathcal{S}^-$ at A.

Corollary 2.12. *The convergence $\mathcal{S}^-$ is topological if and only if*

(Δ^-) $\forall A \subseteq X, \forall \mathbf{U} \in \mathcal{B}_{\mathcal{S}^-}(A), \exists \mathbf{V} \in \mathcal{B}_{\mathcal{S}^-}(A), \forall B \in \mathbf{V}, \exists \mathbf{W} \in \mathcal{B}_{\mathcal{S}^-}(B), \mathbf{W} \subseteq \mathbf{U}$.

Proof. Condition (Δ^-) is just the translation, in terms of neighbourhood bases, of the iterated limit condition (see [17], [13, p. 153]). Thus the corollary follows from the theorem. □

The following corollary follows immediately from Lemma 2.10.

Corollary 2.13. *If the pretopological uniform structure $\mathcal{S}^-$ is quasi-uniform, the convergence $\mathcal{S}^-$ is topological.*

The reverse implication is not true in general.

Example 2.14. Let X be the space of real numbers and $\mathcal{S}$ the family of all finite subsets of X. Then $\mathcal{S}^-$ is topological by Proposition 2.1. We show that $\mathbb{S}^- \nsubseteq \mathbb{S}^- \circ \mathbb{S}^-$. Take $y_1, y_2 \in X$ such that $|y_1 - y_2| > \varepsilon > 0$ and put $S = \{y_1, y_2\}$. Then $\mathbf{S}^-(S;\varepsilon) \in \mathbb{S}^-$. Let $\mathbf{V} = \mathbf{S}^-(S_1, \dots, S_n; \delta) \in \mathbb{S}^-$ be arbitrary. Since S and S_i $(i = 1, \dots, n)$ are finite, there are $c_1, c_2 \in X \setminus (S \cup S_1 \cup \dots \cup S_n)$ such that $|c_1 - y_1| < \delta$ and $|c_2 - y_2| < \delta$. Put $A = S$, $B = \{y_1\}$ and $C = \{c_1, c_2\}$. Then $A \cap S_i \subseteq \{y_1, y_2\} \subseteq C^\delta$ and $C \cap S_i = \emptyset \subseteq B^\delta$. Thus $(A, B) \in \mathbf{V} \circ \mathbf{V}$ but $(A, B) \notin \mathbf{S}^-(S;\varepsilon)$ because $A \cap S = \{y_1, y_2\} \nsubseteq B^\varepsilon$.

The following is a sufficient condition for $\mathbb{S}^-$ to be quasi-uniform.

Proposition 2.15. *Assume that*

(ε) $\forall S \in \mathcal{S}$, $\exists \varepsilon > 0$ *and* $S' \in \mathcal{S}$ *such that* $S^\varepsilon \subseteq S'$.

Then $\mathbb{S}^-$ *is a quasi-uniformity. Consequently,* $\mathcal{S}^-$ *is topological.*

Proof. It is enough to show that $\mathbb{S}^- \subseteq \mathbb{S}^- \circ \mathbb{S}^-$. Take an arbitrary $\mathbf{S}^-(S_1, \dots, S_n; \varepsilon) \in \mathbb{S}^-$. Consider $\varepsilon_1, \dots, \varepsilon_n > 0$ and $S'_1, \dots, S'_n \in \mathcal{S}$ such that $S_i^{\varepsilon_i} \subseteq S'_i$ $(i = 1, \dots, n)$ and pick σ, $0 < \sigma < \min(\varepsilon/2, \varepsilon_1, \dots, \varepsilon_n)$. Then $\mathbf{V} = \mathbf{S}^-(S'_1, \dots, S'_n; \sigma) \in \mathbb{S}^-$. If $(A, B) \in \mathbf{V} \circ \mathbf{V}$ then there is $C \subseteq X$ such that $A \cap S'_i \subseteq C^\sigma$ and $C \cap S'_i \subseteq B^\sigma$ for $i = 1, \dots, n$. We have to show that $A \cap S_i \subseteq B^\varepsilon$ for $i = 1, \dots, n$. If x belongs to $A \cap S_i$ then $x \in A \cap S_i^{\varepsilon_i} \subseteq A \cap S'_i \subseteq C^\sigma$. Hence there is c from C such that $d(c, x) < \sigma$. Consequently, $c \in (A \cap S'_i)^\sigma \subseteq S_i^{\varepsilon_i} \subseteq S'_i$ and $c \in C \cap S'_i \subseteq B^\sigma$. Finally, $x \in (B^\sigma)^\sigma \subseteq B^\varepsilon$. □

Under some additional assumptions on $\mathcal{S}$, the condition (ε) is also necessary.

Proposition 2.16. *Suppose that* $\mathcal{S}$ *is a cover and* $\mathcal{S} = \Sigma(\mathcal{S})$. *If* $\mathbb{S}^-$ *is quasi-uniform, then* $\mathcal{S}$ *has the property* (ε).

Proof. We can assume that $\mathrm{card}(X) > 1$. Now suppose that condition (ε) is not satisfied. Then there is $S_0 \in \mathcal{S}$ such that

$$\text{for every } \varepsilon > 0 \text{ and every } S' \in \mathcal{S},\ S_0^\varepsilon \nsubseteq S'.$$

Because $\mathcal{S}$ is a cover and $\mathcal{S} = \Sigma(\mathcal{S})$ we can assume that $\mathrm{card}(S_0) > 1$. Take two different elements s and r from S_0 and pick δ, $0 < \delta < (1/2)d(s, r)$. We will show that for every $\varepsilon > 0$ and $S_1, \dots, S_n \in \mathcal{S}$,

$$\mathbf{S}^-(S_1, \dots, S_n; \varepsilon) \circ \mathbf{S}^-(S_1, \dots, S_n; \varepsilon) \nsubseteq \mathbf{S}^-(S_0; \delta).$$

Take arbitrary $\varepsilon > 0$ and $S_1, \dots, S_n \in \mathcal{S}$. Then $S_0^\sigma \nsubseteq S = S_1 \cup \dots \cup S_n$ for any σ, $0 < \sigma < \min(\delta, \varepsilon)$. Pick any $x \in S_0^\sigma \setminus S$. Then there is $s_0 \in S_0$ such that $d(s_0, x) < \sigma$. We can assume that $d(s_0, s) > \delta$, otherwise we would take r instead of s. Put $A = \{x, s_0\}$, $B = \{s\}$ and $C = \{x\}$. Then we have $A \cap S_i \subseteq \{s_0\} \subseteq \{x\}^\sigma \subseteq C^\varepsilon$ and $C \cap S_i = \emptyset \subseteq B^\varepsilon$ for $i = 1, 2, \dots, n$. But $(A, B) \notin \mathbf{S}^-(S_0; \delta)$ because $s_0 \in A \cap S \setminus B^\delta$. □

Observe that if $\mathcal{S}$ is a bornology, condition (ε) means that $\mathcal{S}$ is closed with respect to small enlargements:

(E) For every $S \in \mathcal{S}$ there is $\varepsilon > 0$ such that $S^\varepsilon \in \mathcal{S}$.

Thus for bornologies we have the following

Proposition 2.17. *The structure $\mathbb{S}^-$ is quasi-uniform if and only if $\mathcal{S}$ is closed with respect to small enlargements.*

Example 2.18. Let $x_0 \in X$ and consider the cover $\mathcal{S}$ of balls centered at x_0. $\mathcal{S}^-$-convergence is the lower Attouch–Wets convergence AW^- (see [5, p. 81]); since $\mathcal{S}$ verifies the condition in Proposition 2.15, AW^- is topological. As in general, $\mathcal{S}^- = (\downarrow\mathcal{S})^-$, AW^- is equal to $\mathrm{b}(X)^-$, where $\mathrm{b}(X)$ is the family of all bounded subsets of X. By Proposition 2.2, $\mathrm{AW}^- = \mathrm{V}^-$ if and only if each bounded subset of X is totally bounded, and $\mathrm{AW}^- = \mathrm{H}^-$ if and only if X is bounded, by Corollary 2.6(iv).

Remark. All results of this section are valid for uniform spaces.

3. $\mathcal{S}^+$-convergences

As with $\mathcal{S}^-$-convergences, we begin with a few observations:

$$X \in \mathcal{S}^+\text{-}\lim A_t \quad \text{for every } \mathcal{S} \text{ and every net } (A_t), \tag{3.1}$$

$$\text{if } A \in \mathcal{S}^+\text{-}\lim A_t \text{ and } B \supseteq A, \quad \text{then } B \in \mathcal{S}^+\text{-}\lim A_t, \tag{3.2}$$

$$\mathcal{S}^+ = (\downarrow\mathcal{S})^+ = \big(\Sigma(\mathcal{S})\big)^+ = \big(\Sigma(\downarrow\mathcal{S})\big)^+ = \big(\downarrow\big(\Sigma(\mathcal{S})\big)\big)^+, \tag{3.3}$$

$$\text{if } \mathcal{S}_1 \leqslant \mathcal{S}_2, \quad \text{then } \mathcal{S}_1^+ \leqslant \mathcal{S}_2^+. \tag{3.4}$$

Proposition 3.1. *Let $\mathcal{S}$ be a cover of X*:

(i) $\mathrm{s}(X)^+ \leqslant \mathcal{S}^+ \leqslant \{X\}^+$;
(ii) $\{X\}^+ = (2^X)^+ = \mathrm{H}^+$;
(iii) $A \in \mathrm{s}(X)^+\text{-}\lim A_t$ *if and only if the upper limit* $\mathrm{Ls}^\iota A_t$ *of* (A_t) *with respect to the discrete topology of X is contained in* $\bar{A}$.

Proof. (i) and (ii) are clear as $\mathcal{S}$ is a cover. (iii) Suppose that $A \in \mathrm{s}(X)^+\text{-}\lim A_t$ and $x \in \mathrm{Ls}^\iota A_t = \bigcap_t \bigcup_{r \geqslant t} A_r$; then $\forall \varepsilon > 0$, $\exists t_0$, $\forall t \geqslant t_0$, $A_t \cap \{x\} \subseteq A^\varepsilon$; there exists $r \geqslant t_0$ with $x \in A_r$ and therefore $x \in A^\varepsilon$; thus $\mathrm{Ls}^\iota A_t \subseteq \bar{A}$.

Conversely, suppose that $\mathrm{Ls}^\iota A_t \subseteq \bar{A}$ and fix $p \in X$ and $\varepsilon > 0$; if $p \notin \mathrm{Ls}^\iota A_t$, there exists t such that for all $r \geqslant t$, $p \notin A_r$ and $A_r \cap \{p\} \subseteq A^\varepsilon$; if $p \in \mathrm{Ls}^\iota A_t$, $p \in \bar{A}$ and $A_t \cap \{p\} \subseteq A^\varepsilon$ for every t. $\square$

Remark. Note that a net (x_t) of points of X is $\mathrm{s}(X)^+$-convergent to $x \in X$ if and only if $y \neq x$ implies $y \neq x_t$ eventually; thus the restriction of $\mathrm{s}(X)^+$ to X is the cofinite topology and $\mathrm{s}(X)^+$ is not admissible in general. It turns out that the only admissible $\mathcal{S}^+$-convergence is H^+. Of course, H^+ is admissible. If $\mathrm{card}(X) = 1$ then $\mathcal{S}^+ = \mathrm{H}^+$. Assume now that $\mathrm{card}(X) > 1$, $\mathcal{S}^+$ is admissible and suppose that $\mathcal{S}$ does not contain the set X. By (3.3) we can assume that $\mathcal{S} = \Sigma(\mathcal{S})$. Thus we can direct $\mathcal{S}$ upwardly by inclusion and pick $x_S \in X \setminus S$ for each $S \in \mathcal{S}$. Then the net $(\{x_S\})$ is $\mathcal{S}^+$-convergent to every subset of X. Taking any $x, y \in X$ we infer from admissibility of $\mathcal{S}^+$ that (x_S) converges to x and y, i.e., $x = y$. Consequently, $\mathrm{card}(X) = 1$, a contradiction.

We now determine when two covers generate the same convergence.

Theorem 3.2. *Let $\mathcal{S}$ and $\mathcal{W}$ be covers of X. The following conditions are equivalent*:

(i) $\mathcal{S} \leqslant \Sigma(\mathcal{W})$;
(ii) $\mathcal{S}^+ \leqslant \mathcal{W}^+$.

Proof. (i) ⇒ (ii) follows from (3.3) and (3.4).

(ii) ⇒ (i) Suppose $\mathcal{S}$ does not refine $\Sigma(\mathcal{W})$; then there exists $S \in \mathcal{S}$ such that $S \nsubseteq W_1 \cup \cdots \cup W_n$ for every finite union of elements of $\mathcal{W}$; for every $t = (W_1, \ldots, W_n)$ pick $x_t \in S \setminus \bigcup_1^n W_i$. If $t = (W_1, \ldots, W_n)$ and $v = (W_1', \ldots, W_m')$, put $t \leqslant v$ if and only if $W_1 \cup \cdots \cup W_n \subseteq W_1' \cup \cdots \cup W_m'$; it is then easy to see that the net $(\{x_t\})$ is $\mathcal{W}^+$-convergent—but not $\mathcal{S}^+$-convergent—to the empty set. □

We remark that the empty set is isolated for $\mathcal{S}^+$ if and only if $X \in \mathcal{S}$, that is if $\mathcal{S}^+ = \mathrm{H}^+$.

Corollary 3.3. *Let $\mathcal{S}$ and $\mathcal{W}$ be covers of X.*

(i) $\mathcal{S}^+ = \mathcal{W}^+$ *if and only if* $\downarrow\Sigma(\mathcal{S}) = \downarrow\Sigma(\mathcal{W})$;
(ii) *If* $\mathcal{S}^+ \leqslant \mathcal{W}^+$, *then* $\mathcal{S}^- \leqslant \mathcal{W}^-$;
(iii) $\mathcal{S}^+ = \mathrm{s}(X)^+$ *if and only if every* $S \in \mathcal{S}$ *is finite*;
(iv) $\mathcal{S}^+ = \mathrm{H}^+$ *if and only if there exist* $S_1, \ldots, S_k \in \mathcal{S}$ *such that* $X = S_1 \cup \cdots \cup S_k$;
(v) $\mathrm{s}(X)^+ = \mathrm{H}^+$ *if and only if X is finite*;
(vi) *There are noncomparable $\mathcal{S}^+$-convergences.*

Theorem 3.2 and Corollary 3.3(i) are in marked contrast with Theorem 2.5 and Corollary 2.6(i).

Also the example given before Definition 2.4 shows that, in general, $\mathcal{S}^- \leqslant \mathcal{W}^-$ does not imply $\mathcal{S}^+ \leqslant \mathcal{W}^+$.

We will show later (see Proposition 3.6) that $\mathcal{S}^+$-convergence is not topological in general. However, conditions which ensure that $\mathcal{S}^+$-convergence is topological are similar to those for $\mathcal{S}^-$-convergence.

Using the same notation as in Section 2, let us consider the pretopological uniform structure $\mathfrak{S}^+$ on $2^X \times 2^X$ generated by the family $\{\mathbf{S}^+(S_1, \ldots, S_n; \varepsilon)\colon S_1, \ldots, S_n \in \mathcal{S}$ and $\varepsilon > 0\}$, where

$$\mathbf{S}^+(S_1, \ldots, S_n; \varepsilon) = \{(A, B) \in 2^X \times 2^X\colon B \cap S_i \subseteq A^\varepsilon \text{ for } i = 1, \ldots, n\}.$$

Reasoning as in Section 2 we can prove the following

Theorem 3.4. *The pretopological uniform structure $\mathfrak{S}^+$ is compatible with the convergence $\mathcal{S}^+$. Consequently, $\mathcal{S}^+$ is a pretopology on 2^X and for each $A \subseteq X$ the family*

$$\mathcal{B}_{\mathcal{S}^+}(A) = \{\mathbf{S}^+(S_1, \ldots, S_n; \varepsilon)(A)\colon S_1, \ldots, S_n \in \mathcal{S} \text{ and } \varepsilon > 0\}$$

is a local base of $\mathcal{S}^+$ at A.

Since $\mathcal{S}^+$ is a pretopology, it is topological if and only if it verifies the condition

(Δ^+) $\forall A \subseteq X, \forall \mathbf{U} \in \mathcal{B}_{\mathcal{S}^+}(A), \exists \mathbf{V} \in \mathcal{B}_{\mathcal{S}^+}(A), \forall B \in \mathbf{V}, \exists \mathbf{W} \in \mathcal{B}_{\mathcal{S}^+}(B), \mathbf{W} \subseteq \mathbf{U}.$

The condition (ε) from Proposition 2.15 works also for the structure $\mathfrak{S}^+$.

Proposition 3.5. *Suppose that $\mathcal{S}$ is a cover and $\mathcal{S} = \Sigma(\mathcal{S})$. Then $\mathfrak{S}^+$ is a quasi-uniformity if and only if*

(ε) $\forall S \in \mathcal{S}, \exists \varepsilon > 0$ *and* $S' \in \mathcal{S}$ *such that* $S^\varepsilon \subseteq S'$.

Although there are many similarities between the $\mathcal{S}^-$ and $\mathcal{S}^+$-convergences, there are also important differences. One of them is the behaviour of the convergence $\mathrm{s}(X)^+$. In contrast with $\mathrm{s}(X)^-$, $\mathrm{s}(X)^+$ is not topological in general.

Proposition 3.6. *The convergence $\mathrm{s}(X)^+$ is topological if and only if (X, d) is discrete.*

Proof. If (X, d) is discrete then $\mathrm{s}(X)^+$ is topological by Corollary 3.3(iii) and Proposition 3.5. The necessary condition follows from Theorem 3.7 below. □

Theorem 3.7. *Assume that $\mathcal{S} = \Sigma(\mathcal{S})$. If $\mathcal{S}^+$ is topological then every nondense subset $S \in \mathcal{S}$ has the property*

(ε') $\exists \varepsilon > 0$ *and* $S' \in \mathcal{S}$ *such that* $S^\varepsilon \subseteq S'$.

Proof. Suppose that there is $S_0 \in \mathcal{S}$ such that $\bar{S}_0 \neq X$ and $S_0^\varepsilon \nsubseteq S$ for every $\varepsilon > 0$ and every $S \in \mathcal{S}$. We shall show that condition (Δ^+) does not hold. Let us take $x_0 \in X$ and $\varepsilon_0 > 0$ such that $\{x_0\}^{\varepsilon_0} \cap S_0 = \emptyset$. Now put $A = \{x_0\}$ and consider the neighbourhood

$$\mathbf{U} = \{D \subseteq X\colon D \cap S_0 \subseteq A^\varepsilon\} \in \mathcal{B}_{\mathcal{S}^+}(A).$$

Take an arbitrary $\mathbf{V} \in \mathcal{B}_{\mathcal{S}^+}(A)$ of the form

$$\mathbf{V} = \{E \subseteq X\colon E \cap S_i \subseteq A^\sigma \text{ for } i = 1, \ldots, k\}.$$

Then $S_0^{1/n} \nsubseteq S = S_1 \cup \cdots \cup S_k$ for every $n \in \mathbb{N}$. Pick $x_n \in S_0^{1/n} \setminus S$ for every $n \in \mathbb{N}$. Then the set $B = \{x_n\colon n \in \mathbb{N}\}$ belongs to $\mathbf{V}$. Now let

$$\mathbf{W} = \{C \subseteq X\colon\ C \cap S_i' \subseteq B^\delta \text{ for } i = 1, \ldots, m\} \in \mathfrak{B}_{\mathcal{S}^+}(B)$$

be arbitrary. Take $n \in \mathbb{N}$ such that $1/n < \delta$ and $y \in B^{1/n} \cap S_0$. Then $C = \{y\} \in \mathbf{W}$ but $C \notin \mathbf{U}$ because $C \cap S_0 \nsubseteq A^\varepsilon$. □

Corollary 3.8. *Suppose that $\mathcal{S} = \Sigma(\mathcal{S})$ and $\mathcal{S} \leqslant \mathcal{S} \cap \mathrm{cl}(X)$.*

(i) *$\mathcal{S}^+$ is topological if and only if condition (ε) is verified*;
(ii) *If $\mathcal{S}^+$ is topological then $\mathcal{S}^-$ is topological*;
(iii) *$(2^X, \mathcal{S}^+)$ is topological if and only if $(2^X, \mathcal{S}^-, \mathcal{S}^+)$ is bitopological.*

Observe that if $\mathcal{S} = {\downarrow}\mathcal{S}$ then the inequality $\mathcal{S} \leqslant \mathcal{S} \cap \mathrm{cl}(X)$ is equivalent to the property that $\mathcal{S}$ is closed with respect to the closure operator $\mathcal{S} \ni S \mapsto \bar{S} \in \mathcal{S}$. Applying Corollary 3.8 we infer that if a bornology $\mathcal{S}$ is closed with respect to the closure operator then $\mathcal{S}^+$ is topological if and only if the structures $\mathfrak{S}^-$ and $\mathfrak{S}^+$ are quasi-uniformities.

Notice that the converse of Corollary 3.8(ii) is not true in general (see, e.g., Proposition 3.6).

Recall that $\mathrm{c}(X)$ and $\mathrm{tb}(X)$ are the families of all compact and totally bounded subsets of (X, d), respectively.

Corollary 3.9.

(i) $\mathrm{c}(X)^+$ *is topological if and only if X is locally compact*;
(ii) $\mathrm{tb}(X)^+$ *is topological if and only if X is locally totally bounded.*

Proof. (i) The sufficient condition follows from Proposition 3.5. Now assume that $\mathrm{c}(X)^+$ is topological and take any $x \in X$. If $\mathrm{card}(X) > 1$, then $\{x\}$ is not dense in X. Since $\{x\} \in \mathrm{c}(X)$, it follows from Theorem 3.7 that $\{x\}^\varepsilon$ is contained in a compact set for some $\varepsilon > 0$.

(ii) The proof is analogous. □

We now look at the relationship between $\mathcal{S}^+$-convergences and some other "upper" convergences. Following Beer [5, p. 44] we will consider the miss topology $\mu_{\mathcal{S}}$ on 2^X determined by $\mathcal{S}$. This topology has as a subbase all sets of the form $\{A \subseteq X\colon\ A \subseteq S^c\}$, where $S \in \mathcal{S}$.

We say that a subset A of X strongly misses $\mathcal{S}$, and write $A \ll \mathcal{S}$, if for each $S \in \mathcal{S}$, $A \cap S = \emptyset$ implies $A^\varepsilon \cap S = \emptyset$ for some $\varepsilon > 0$. For $\mathcal{A} \subseteq 2^X$ we write $\mathcal{A} \ll \mathcal{S}$ if $A \ll \mathcal{S}$ for each $A \in \mathcal{A}$.

We will write $\mathcal{S} = {\downarrow_{\mathrm{cl}}}\mathcal{S}$ if $\mathcal{S}$ contains all sets of the form $S \cap C$, where $S \in \mathcal{S}$ and $C \in \mathrm{cl}(X)$. Notice that $\mathcal{S} = {\downarrow}\mathcal{S}$ implies $\mathcal{S} = {\downarrow_{\mathrm{cl}}}\mathcal{S}$.

Proposition 3.10.

(i) *If $\mathcal{S} = {\downarrow_{\mathrm{cl}}}\mathcal{S}$ then $\mathcal{S}^+ \leqslant \mu_{\mathcal{S}}$*;
(ii) *$A \in \mathcal{S}^+\text{-}\lim A_t$ implies $A \in \mu_{\mathcal{S}}\text{-}\lim A_t$ if and only if $A \ll \mathcal{S}$.*

Proof. (i) Suppose $A_t \to A$ with respect to $\mu_{\mathcal{S}}$ and fix $S \in \mathcal{S}$, $\varepsilon > 0$; consider $S_1 = S \cap (A^\varepsilon)^c \in \mathcal{S}$; then $A \cap S_1 = \emptyset$ and thus $A_t \cap S_1 = \emptyset$ eventually. This implies that $A_t \cap S \subseteq A^\varepsilon$ eventually.

(ii) $\Rightarrow$ Suppose that there is $S_0 \in \mathcal{S}$ such that $A \cap S_0 = \emptyset$ and $A^\varepsilon \cap S_0 \neq \emptyset$ for each $\varepsilon > 0$. Put $A_n = A^{1/n}$. Then $A \in \mathcal{S}^+$-lim A_n but (A_n) does not $\mu_{\mathcal{S}}$-converge to A.

(ii) $\Leftarrow$ Suppose $A_t \to A$ with respect to $\mathcal{S}^+$ and $A \cap S = \emptyset$; then $A^\varepsilon \cap S = \emptyset$ for some $\varepsilon > 0$ and $A_t \cap S \subseteq A^\varepsilon$ eventually; thus $A_t \cap S = \emptyset$ eventually. □

Corollary 3.11.

(i) *If* $\mathcal{S} = {\downarrow_{\mathrm{cl}}}\mathcal{S}$ *then* $\mathcal{S}^+ = \mu_{\mathcal{S}}$ *on* $\mathcal{A}$ *if and only if* $\mathcal{A} \ll \mathcal{S}$;
(ii) *If* $\mathcal{S} = {\downarrow_{\mathrm{cl}}}\mathcal{S}$ *and* $\mathcal{A} \ll \mathcal{S}$ *then* $\mathcal{S}^+$ *restricted to* $\mathcal{A}$ *is topological*;
(iii) *The convergences* $\mathrm{s}(X)^+$ *and* $\mathrm{c}(X)^+$ *restricted to* $\mathrm{cl}(X)$ *are topological*;
(iv) *The convergence* $\mathrm{tb}(X)^+$ *restricted to* $\mathrm{c}(X)$ *is topological.*

It follows from Corollary 3.11(i) that not every topological $\mathcal{S}^+$-convergence is equal to the topology $\mu_{\mathcal{S}}$ on $\mathrm{cl}(X)$. Indeed, $\mathrm{b}(X)^+$ is topological by Proposition 3.5 but $\mathrm{cl}(X)$ does not strongly miss $\mathrm{b}(X)$ in general.

From Corollary 3.11 we infer also that $\mathrm{s}(X)^+$ amounts to the co-finite and $\mathrm{c}(X)^+$ to the co-compact topology on $\mathrm{cl}(X)$. This leads to the

Corollary 3.12 (cf. [5, Theorem 5.1.6]). *If* $\mathcal{S}$ *is the family of all compact subsets of* X *then the* $\mathcal{S}^+$*-convergence coincides with the co-compact topology on* $\mathrm{cl}(X)$.

It is well known (see, e.g., [17, p. 43]) that H^+-convergence implies the upper Kuratowski convergence K^+ on $\mathrm{cl}(X)$. So it is natural to ask which $\mathcal{S}^+$-convergences imply the convergence K^+. Recall that $A_t \to A$ for K^+ if $\mathrm{Ls}\, A_t \subseteq A$, where $\mathrm{Ls}\, A_t = \bigcap_t \overline{\bigcup_{s \geqslant t} A_s}$. We begin with the following

Proposition 3.13.

(i) *If a net* (A_t) *is* $\mathcal{S}^+$*-convergent to* A*, then for every* $S \in \mathcal{S}$, $\mathrm{Ls}(A_t \cap S) \subseteq \bar{A}$;
(ii) *Let* $\mathcal{S}$ *be the family of all compact subsets of* X*. Then a net* (A_t) *is* $\mathcal{S}^+$*-convergent to* A *if and only if for every* $S \in \mathcal{S}$, $\mathrm{Ls}(A_t \cap S) \subseteq \bar{A}$.

Proof. (i) Let $S \in \mathcal{S}$ and suppose that $x \in \mathrm{Ls}(A_t \cap S)$; fix $\varepsilon > 0$; there exists t_0 such that for all $t \geqslant t_0$, $A_t \cap S \subseteq A^{\varepsilon/2}$, and for some $t \geqslant t_0$, $B(x, \varepsilon/2) \cap A_t \cap S \neq \emptyset$; thus $x \in A^\varepsilon$ for all $\varepsilon > 0$ and $x \in \bar{A}$.

(ii) Apply (i), Proposition 3.10 and [5, Proposition 5.2.5]. □

It follows from (ii) of the above proposition that $\mathrm{K}^+ \geqslant \mathrm{c}(X)^+$ on $\mathrm{cl}(X)$.

The following lemma was pointed out to us by G. Beer.

Lemma 3.14. *If* $\mathcal{S} = {\downarrow}\mathcal{S}$ *then the following conditions are equivalent*:

(i) $\forall x \in X$, $\exists \delta > 0$ *such that* $\{x\}^\delta \in \mathcal{S}$;
(ii) *For every net* (A_t) *which is* $\mathcal{S}^+$*-convergent to* A, *we have* $\mathrm{Ls}\, A_t \subseteq \bar{A}$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in \mathrm{Ls}\, A_t$ and take $\varepsilon > 0$. Pick $0 < \delta < \varepsilon/2$ such that $\{x\}^\delta \in \mathcal{S}$ and proceed as in the proof of Proposition 3.13(i).

(ii) $\Rightarrow$ (i) Suppose there is $x_0 \in X$ such that $\{x_0\}^{1/n} \notin \mathcal{S}$ for every $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ and $S \in \mathcal{S}$ there is $x_{(n,S)} \in \{x_0\}^{1/n} \setminus S$. It is clear that the net $(\{x_{(n,S)}\})$ is $\mathcal{S}^+$-convergent to the empty set, while $x_0 \in \mathrm{Ls}\{x_{(n,S)}\}$. □

As a counterpart to the preceding lemma, we have

Lemma 3.15. *If* K^+ *is stronger than* $\mathcal{S}^+$, *every* $S \in \mathcal{S}$ *is relatively compact.*

Proof. Suppose there exists $S \in \mathcal{S}$ which is not relatively compact; let (x_n) be a sequence in S without converging subsequence and put $A_n = \{x_m\colon m > n\}$. Then $\mathrm{Ls}\, A_n = \emptyset$ but (A_n) is not $\mathcal{S}^+$-convergent to the empty set as $A_n \cap S$ is not empty for every n. □

An application of the two previous lemmas yields

Corollary 3.16. $\mathrm{K}^+ = \mathcal{S}^+$ *if and only if* X *is locally compact and* $\mathcal{S}$ *generates the bornology of relatively compact subsets* (*equivalently,* $\mathcal{S}^+$ *is the co-compact topology*).

Let us now consider the bornology $\mathrm{tb}(X)$ of all totally bounded subsets of (X, d). Of course, $\mathrm{c}(X)^+ \leqslant \mathrm{tb}(X)^+$.

Theorem 3.17. *The following conditions are equivalent*:

(i) *The metric* d *is complete*;
(ii) $\mathrm{tb}(X)^+ = \mathrm{c}(X)^+$;
(iii) $\mathrm{tb}(X)^+ \leqslant \mathrm{K}^+$ *on* $\mathrm{cl}(X)$.

Proof. (i) $\Rightarrow$ (ii) If d is complete then $\mathrm{tb}(X) \leqslant \mathrm{c}(X)$ and (3.4) implies (ii).

(ii) $\Rightarrow$ (iii) is obvious.

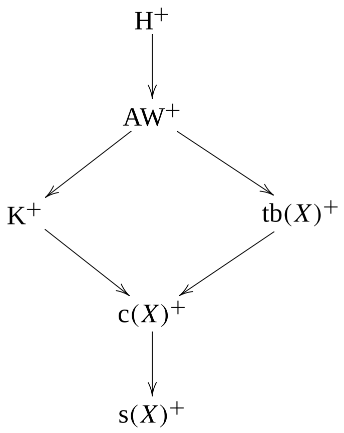
(iii) $\Rightarrow$ (i) By Lemma 3.15 each totally bounded subset of X is relatively compact; hence d is complete. □

By Lemma 3.14, $\mathrm{K}^+ \leqslant \mathrm{tb}(X)^+$ on $\mathrm{cl}(X)$ if and only if (X, d) is locally totally bounded. Thus we have

Corollary 3.18. $\mathrm{K}^+ = \mathrm{tb}(X)^+$ *on* $\mathrm{cl}(X)$ *if and only if* (X, d) *is locally compact and complete.*

We end this section with a reference to $\mathrm{b}(X)^+$, where $\mathrm{b}(X)$ is the family of all bounded subsets of X. $\mathrm{b}(X)^+$ is topological by Proposition 3.5 and it is equal to the upper Attouch–Wets convergence AW^+, as described in [5, p. 81]. By Corollary 3.3, $\mathrm{AW}^+ = \mathrm{s}(X)^+$ if and only if each bounded subset of X is finite and $\mathrm{AW}^+ = \mathrm{H}^+$ if and only if X is bounded.

The relationship of the main $\mathcal{S}^+$-convergences on $\mathrm{cl}(X)$ is expressed by the following diagram:



Remark. All results of this section are valid for uniform-spaces.

4. $\tilde{\mathcal{S}}$-convergences

We start with a result linking $\tilde{\mathcal{S}}$ and $\mathcal{S}^+$ convergences:

Lemma 4.1. *A net (A_t) is $\mathcal{S}^+$-convergent to A if and only if the net $(A_t \cup A)$ is $\tilde{\mathcal{S}}$-convergent to A.*

Proof. $\Rightarrow$ If (A_t) is $\mathcal{S}^+$-convergent to A, given S and $\varepsilon > 0$, $(A_t \cup A) \cap S = (A_t \cap S) \cup (A \cap S) \subseteq A^\varepsilon$ eventually; also it is clear that $(A_t \cup A)$ is $\mathcal{S}^-$-convergent to A.

$\Leftarrow$ If $(A_t \cup A)$ is $\tilde{\mathcal{S}}$-convergent to A, $(A_t \cup A)$ is $\mathcal{S}^+$-convergent to A and $(A_t \cup A) \cap S \subseteq A^\varepsilon$ eventually, so that $A_t \cap S \subseteq A^\varepsilon$ eventually. $\square$

Corollary 4.2. *Let $\mathcal{S}$ and $\mathcal{W}$ be covers of X. Then*

(i) $\tilde{\mathcal{S}} \leqslant \tilde{\mathcal{W}}$ *if and only if* $\mathcal{S}^+ \leqslant \mathcal{W}^+$;
(ii) $\tilde{\mathcal{S}} = \tilde{\mathcal{W}}$ *if and only if* $\mathcal{S}^+ = \mathcal{W}^+$ *and this implies* $\mathcal{S}^- = \mathcal{W}^-$.

Proof. If $\tilde{\mathcal{S}} \leqslant \tilde{\mathcal{W}}$ and $A_t \stackrel{\mathcal{S}^+}{\to} A$ then, by Lemma 4.1, $A_t \cup A \stackrel{\tilde{\mathcal{W}}}{\to} A$. Applying again Lemma 4.1 we get $A_t \to A$ with respect to $\mathcal{W}^+$. The necessary condition follows from Corollary 3.3(ii).

(ii) follows from (i). $\square$

Remark. Note that the mapping $x \mapsto \{x\}$ is continuous with respect to $\tilde{\mathcal{S}}$. Thus $\tilde{\mathcal{S}}$ is admissible because it is finer than $\mathcal{S}^-$ which is admissible.

Now we come to the question of uniform properties of $\tilde{\mathcal{S}}$. For $S_1, \ldots, S_n \in \mathcal{S}$ and $\varepsilon > 0$ put

$$\mathbf{S}(S_1, \ldots, S_n; \varepsilon) = \{(A, B) \in 2^X \times 2^X\colon\ A \cap S_i \subseteq B^\varepsilon,\ B \cap S_i \subseteq A^\varepsilon,\ i = 1, \ldots, n\}.$$

Then the family $\{\mathbf{S}(S_1, \ldots, S_n; \varepsilon)\colon\ S_1, \ldots, S_n \in \mathcal{S}$ and $\varepsilon > 0\}$ is a base of the pretopological uniform structure $\tilde{\mathbb{S}} = \mathbb{S}^- \vee \mathbb{S}^+$.

From Theorems 2.11 and 3.4 we infer

Theorem 4.3. *The pretopological uniform structure $\tilde{\mathbb{S}}$ is compatible with the convergence $\tilde{\mathcal{S}}$. Consequently, $\tilde{\mathcal{S}}$ is a pretopology on 2^X and for each $A \subseteq X$ the family*

$$B_{\tilde{\mathcal{S}}}(A) = \{\mathbf{S}(S_1, \ldots, S_n; \varepsilon)(A)\colon\ S_1, \ldots, S_n \in \mathcal{S} \text{ and } \varepsilon > 0\}$$

is a local base of $\tilde{\mathcal{S}}$ at A.

Of course, $\tilde{\mathcal{S}} = \mathcal{S}^- \vee \mathcal{S}^+$ and $\tilde{\mathcal{S}}$ is topological if and only if it verifies the condition

(Δ) $\forall A \subseteq X, \forall \mathbf{U} \in \mathcal{B}_{\tilde{\mathcal{S}}}(A), \exists \mathbf{V} \in \mathcal{B}_{\tilde{\mathcal{S}}}(A), \forall B \in \mathbf{V}, \exists \mathbf{W} \in \mathcal{B}_{\tilde{\mathcal{S}}}(B), \mathbf{W} \subseteq \mathbf{U}$.

Combining Propositions 2.15 and 3.5 we obtain

Proposition 4.4. *Suppose that $\mathcal{S}$ is a cover and $\mathcal{S} = \Sigma(\mathcal{S})$. Then $\tilde{\mathbb{S}}$ is a uniformity if and only if*

(ε) $\forall S \in \mathcal{S}, \exists \varepsilon > 0$ *and* $S' \in \mathcal{S}$ *such that* $S^\varepsilon \subseteq S'$.

Remark. It follows from Proposition 4.4 and Corollary 3.8 that if a bornology $\mathcal{S}$ is closed with respect to the closure operator, then $\mathcal{S}^+$ is topological if and only if $\tilde{\mathbb{S}}$ is a uniformity.

We will now look at the separation properties of $\tilde{\mathcal{S}}$-convergences.

Recall that a pretopological space (Z, π) is T_1 if $\{z\}$ is closed for each $z \in Z$, T_2 or *Hausdorff* if every two distinct elements of Z have disjoint neighbourhoods, and, is T_3 or *regular* if for each $z \in Z$ and $U \in \mathcal{N}_\pi(z)$ there is $V \in \mathcal{N}_\pi(z)$ such that $\bar{V} \subseteq U$.

Since $\tilde{\mathcal{S}}$ is a pretopology, it is T_1 if and only if $\bigcap \tilde{\mathbb{S}}$ is contained in the diagonal of $2^X \times 2^X$. Consequently, $(2^X, \tilde{\mathcal{S}})$ is not T_1 in general. Indeed, take $A \subseteq X$ such that $\bar{A} \neq A$. Then $(\bar{A}, A)$ belongs to every element of $\tilde{\mathbb{S}}$.

Proposition 4.5.

(i) $(\mathrm{cl}(X), \tilde{\mathcal{S}})$ *is T_1 if and only if $\mathcal{S}$ is a cover of X;*
(ii) *Let $\mathcal{S} = \downarrow\mathcal{S}$. Then $(\mathrm{cl}(X), \tilde{\mathcal{S}})$ is T_2 if and only if the condition* (i) *in Lemma* 3.14 *is verified.*

Proof. (i) $\Rightarrow$ Suppose that there is $x \in X$ such that $x \notin S$ for every $S \in \mathcal{S}$. Then $(\{x\}, \emptyset)$ belongs to every element of element of $\tilde{\mathbb{S}}$. A contradiction.

$\Leftarrow$ An easy proof is omitted.

(ii) $\Rightarrow$ Suppose there exists $x \in X$ that does not verify the condition; then for every n and every $S \in \mathcal{S}$ there exists $x_{(n,S)} \in \{x\}^{1/n} \setminus S$. It can be checked that the net $(\{x_{(n,S)}\})$ $\tilde{\mathcal{S}}$-converges to both $\{x\}$ and $\emptyset$; therefore the $\tilde{\mathcal{S}}$-convergence is not Hausdorff.

$\Leftarrow$ Let A and B be closed and $A \neq B$. We can assume that there is $x \in A \setminus B$. Then $\{x\}^\varepsilon \cap B = \emptyset$ for some $\varepsilon > 0$. Take $0 < \delta < \varepsilon/2$ such that $S = \{x\}^\delta \in \mathcal{S}$. Put $\mathbf{U} = \mathbf{S}^-(S;\delta)(A)$ and $\mathbf{V} = \mathbf{S}^+(S;\delta)(B)$. Then $\mathbf{U} \cap \mathbf{V} = \emptyset$. □

It follows from Propositions 4.5(i) and 4.4 that if a bornology $\mathcal{S}$ is closed with respect to small enlargements then $\tilde{\mathbf{S}}$ is a Hausdorff uniformity on $\mathrm{cl}(X) \times \mathrm{cl}(X)$ and $(\mathrm{cl}(X), \tilde{\mathcal{S}})$ is a completely regular topological space.

Proposition 4.6. *Let $\mathcal{S}$ be a bornology. The following conditions are equivalent*:

(i) *The empty set has a closed local base for $\tilde{\mathcal{S}}$;*
(ii) *For every $S \in \mathcal{S}$, there exists $S' \in \mathcal{S}$ such that $S \subseteq \mathrm{int}(S')$;*
(iii) $\forall A \in \mathrm{cl}(X)$, $\forall S \in \mathcal{S}$, $\forall \varepsilon > 0$, $\exists S' \in \mathcal{S}$, $\overline{\mathbf{S}(S';\varepsilon/2)(A)}^{\tilde{\mathcal{S}}} \subseteq \mathbf{S}^+(S;\varepsilon)(A)$.

Proof. (i) implies (ii). Let S be in $\mathcal{S}$; put $\mathbf{S}^+(S)(\emptyset) = \{B\colon\ B \cap S = \emptyset\}$; then there exists $S' \in \mathcal{S}$ such that $\overline{\mathbf{S}(S')(\emptyset)}^{\tilde{\mathcal{S}}} \subseteq \mathbf{S}^+(S)(\emptyset)$; suppose that $S \not\subset \mathrm{int}(S')$; then there exists x in S which does not belong to $\mathrm{int}(S')$ and we can find a sequence (x_n) in the complement of S' that converges to x. Thus, for every n, $\{x_n\} \in \mathbf{S}^+(S')(\emptyset)$, and $\{x\} = \tilde{\mathcal{S}}\text{-}\lim\{x_n\}$ must be in $\mathbf{S}^+(S)(\emptyset)$, a contradiction.

Note that condition (i) implies that the $\tilde{\mathcal{S}}$-convergence is Hausdorff.

(ii) implies (iii). Let S be in $\mathcal{S}$ and consider $S' \in \mathcal{S}$ given by (ii); take any $D \in \overline{\mathbf{S}(S';\varepsilon/2)(A)}^{\tilde{\mathcal{S}}}$ and let (D_t) be a net of elements in $\mathbf{S}(S';\varepsilon/2)(A)$ which $\tilde{\mathcal{S}}$-converges to D; if $x \in D \cap S$, there is $0 < \sigma < \varepsilon/2$ such that $\{x\}^\sigma \subseteq S'$ and $D \cap \{x\}^\sigma \subseteq D_t^\sigma$ eventually; thus $d(x, d_t) < \sigma$ for some t and some $d_t \in D_t$. Since $D_t \in \mathbf{S}(S';\varepsilon/2)(A)$, we obtain that $d_t \in A^{\varepsilon/2}$ and finally that $x \in A^\varepsilon$.

(iii) implies (i). Apply (iii) to the empty set. □

Remarks. (a) If no $S \in \mathcal{S}$ is dense, a weaker form of (iii) with $A \neq \emptyset$ implies (i).

(b) If $\mathcal{S} \leqslant \mathcal{S} \cap \mathrm{cl}(X)$, (ii) is equivalent to $\forall S \in \mathcal{S}$, $\exists\, S' \in \mathcal{S}$ such that $\bar{S} \subseteq \mathrm{int}(S')$.

Corollary 4.7. *The small enlargement condition for $\mathcal{S}$ implies regularity for* $(\mathrm{cl}(X), \tilde{\mathcal{S}})$*, which, in turn, implies condition* (ii) *above.*

We close the paper with a word on hit-and-miss topologies: if Δ is a subfamily of $\mathrm{cl}(X)$, the hit-and-miss topology determined by Δ is the supremum of the lower Vietoris topology and of the miss topology μ_Δ (see, for instance, [5,8,9,14,21]). Under mild assumptions on the cover $\mathcal{S}$, we have $\mathrm{V}^- \leqslant \mathcal{S}^-$ and $\mathcal{S}^+ \leqslant \mu_{\mathcal{S}}$ by Propositions 2.1(i) and 3.10(i). Thus the $\tilde{\mathcal{S}}$-convergence, which is more "symmetric" in nature, is not comparable, except for the Fell topology, to the corresponding hit-and-miss topology.

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