On set convergences and topologies

Part I - Basics

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What is a set convergence (topology)?

- Let X be a (nonempty) set. A ***set convergence (topology)*** is a convergence (topology) defined on the power set 2^X (or on any subset of 2^X).
- A set convergence Π on 2^X assigns to each net (A_t) (or a filter $\mathcal{F}$) defined in 2^X a subset $\mathrm{Lim}^\Pi A_t$ of 2^X.
- If $A \in \mathrm{Lim}^\Pi A_t$ then we write $A_t \xrightarrow{\Pi} A$ (or just $A_t \to A$).
- The pair $(2^X, \Pi)$ is often called ***hyperspace.***

What is actually new in this idea?

- Since $(2^X, \subseteq)$ is a partially ordered set there is a natural convergence on 2^X: the ***order convergence***
  - But if card $X > 2$, the order convergence on 2^X is the discrete convergence.
- One could then consider set convergences introduced in an axiomatic way
  - But arbitrary set convergences would not be very useful (and would not mean anything new).
- We are rather interested in set convergences Π defined on the power set 2^X (or a subset of 2^X) of a topological space (X, π) which are somehow linked to the underlying topology π.
- What we are interesting in is e.g. the interplay between π and Π.
- Is for example Π "compatible" with π, i.e. is the mapping $x \to \{x\}$ an embedding? (in this case we call Π ***admissible***).
- In contrast to a "usual" topological space elements of a hyperspace (being subsets of the underlying space) can have a much richer structure.

If set convergences are the answer, what are the questions?

- EXAMPLE
  - Consider an optimization problem of the form

    (OP) $\min \{f(x): x \in K\}$

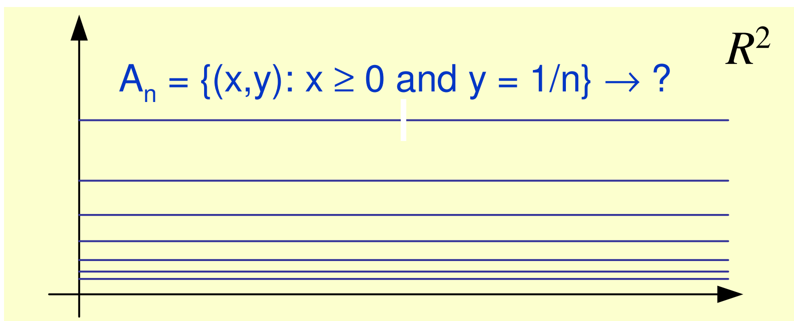
    where $f:X \to \boldsymbol{R}$ and $K \subseteq X$.
  - Usually we have to deal with the parametric optimization problem

    (OP_y) $\min \{f(x,y): x \in K_y\}$,

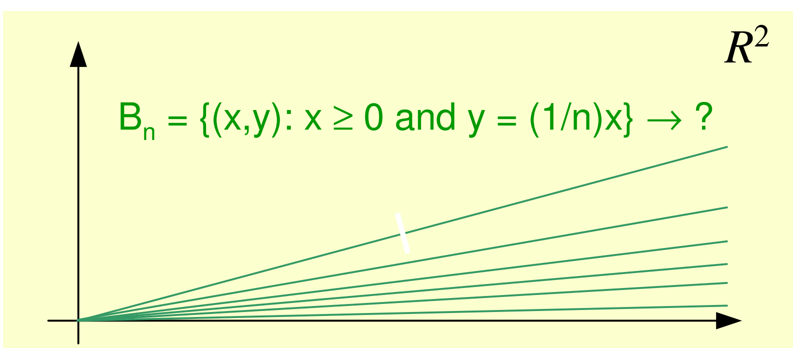
    where $f:X \times Y \to \boldsymbol{R}$ and $K_y \subseteq X$ for $y \in Y$.
  - Stability problem
    - *Is the solution function $y \to v(y) = \min \{f(x,y): x \in K_y\}$ continuous?*
    - *Is the solution set mapping $y \to M(y) = \{x \in K_y: f(x,y) = v(y)\}$ "continuous"?*
  - But what does it mean that $y \to M(y) \subseteq X$ (i.e. $M:Y \to 2^X$) is ***continuous***?

What would be a “good” set convergence?

- Easy to construct?
- Strong or rather weak? Admissible?
- Reflecting (possibly many) properties of the underlying topological space?
- Our “standard tests for goodness” of a set convergence
  - EXAMPLE (Test A):



  - EXAMPLE (Test B):



From a “good” convergence we would expect that both (A_n) and (B_n) converge to the semiline $\{(x,y) : x \geq 0 \text{ and } y = 0\}$

Main questions

- For a given topology π on X how to construct a convergence (topology) on a the power set 2^X (or a subfamily $\subseteq 2^X$)?
- How to construct "good" set convergences?
- Are there many (infinite) constructions leading to meaningful set convergences?
- In other words: how to enter hyperspace (without warp drive)?

Standard constructions (1): hit-convergences

- Usual convergence of nets in a topological space X:

  $x_t \to x$ if for every open set U, $x \in U$ implies that $x_t \in U$ eventually.

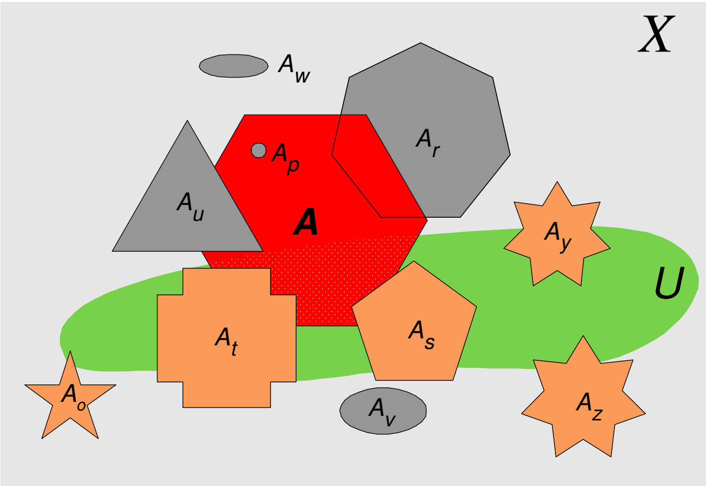
- But $y \in U \Leftrightarrow \{y\} \cap U \neq \emptyset$, thus

  $x_t \to x$ if and only if for every open set U with $\{x\} \cap U \neq \emptyset$ we have $\{x_t\} \cap U \neq \emptyset$ eventually.

- Following this, we can define for a net (A_t) of subsets of X

  $A_t \to A$ if for every open set U,

  $A \cap U \neq \emptyset$ implies that $A_t \cap U \neq \emptyset$ eventually.



- This is an example of a so-called ***hit-convergence***:

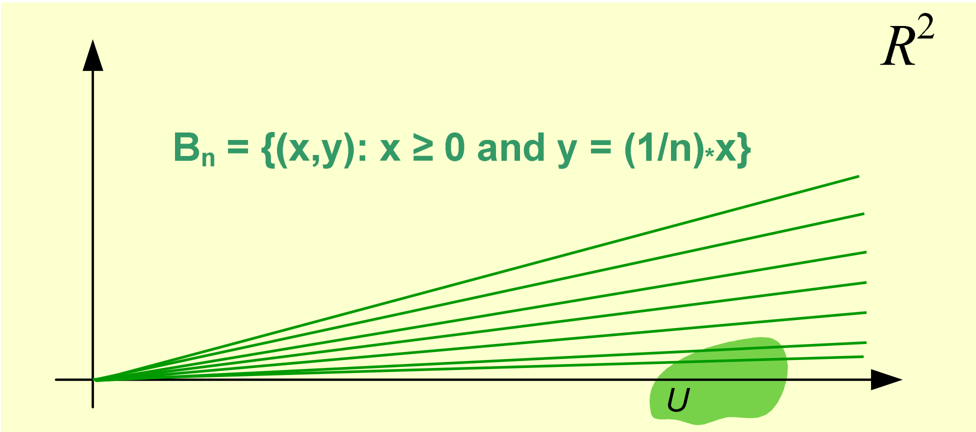
  if A hits an open set U then A_t hits U eventually.

- This convergence is usually called ***lower Vietoris convergence*** V^-.

- V^- is clearly admissible: $x_t \to x$ iff $\{x_t\} \to \{x\}$ with respect to V^-.

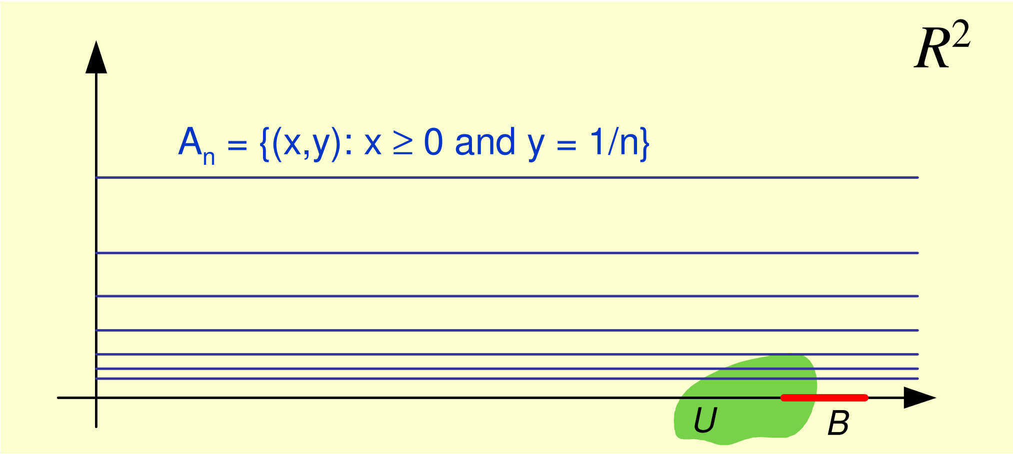
Lower Vietoris convergence: basic properties (1)

- The convergence V^- is actually a topology: the family $\{\#U: U \text{ is open}\}$, where $\#U = \{A \subseteq X: A \cap U \neq \emptyset\}$, is a sub-base of a topology on 2^X compatible with V^-.
- Let $\downarrow B = \{A \subseteq X: A \subseteq B\}$. Then $(\#U)^c = \downarrow(X \setminus U)$ and it means that V^- is the weakest set topology in which all sets of the form $\downarrow F$ are closed, where $F \subseteq X$ is closed.
- Is V^- a "good" convergence?
  - Sequences (A_n) and (B_n) converge obviously to the semiline $\{(x,y) : x \geq 0 \text{ and } y = 0\}$.



Lower Vietoris convergence: basic properties (2)

- However, V^- can be seen as too coarse: if $A_t \to A$ then $A_t \to B$ for ***any*** subset B of A.
- EXAMPLE



- V^- is too coarse for good separation properties: V^- is never T_1 (if card X > 1).
- V^- need not be even T_0
  - Consider ***R*** with the standard topology and take A = (0,1), B = [0,1]. Then $A \neq B$ but A belongs to every V^-–neighborhood of B and B belongs to every V^-–neighborhood of A.

What about other hit-convergences?

- The hit-convergence V^- is generated by the family $\mathcal{O}$ of open subsets of X.
- Are there any other "natural" families that lead to "good" hit-convergences?
- Let us for example consider the hit-convergence Π generated by the family of all closed subsets of X (where X is T_1)
  - Notice that if $A_t \to A$ with respect to Π then $A \subseteq \bigcup_t \bigcap_{s \geq t} A_s$ (because singletons {x} are closed.
  - It means that Π is too strong to be interesting (Π does not pass our "goodness" tests A and B).
  - The same is true for any hit-convergence generated by a family containing all singletons.
- There are only a few know hit-convergences with good properties.

Set convergences: adjusting parameters

- Let (X, π) be a topological space and Π a set convergence on a family $\mathcal{A} \subseteq 2^X$ and let us assume that Π is an "extension" of π.
- Π depends of course on π (i.e. on the way how it was constructed using π).
- But Π depends also on the choice of the subfamily $\mathcal{A}$ on which it is defined.
- EXAMPLE
  The lower Vietoris convergence V^- restricted to CL(X) is T_0:
  if A and B are closed (nonempty) and $A \neq B$ then e.g. A is not included in B. Thus there is $x \in A \setminus B$ and we can find an open neighborhood U of x that is disjoint with B. Consequently $\#U = \{C \subseteq X: C \cap U \neq \emptyset\}$ is a V^--neighborhood of A that does not contain B.
- Choice of the family $\mathcal{A}$ is a tradeoff: $\mathcal{A}$ should be taken large enough to be interesting for applications and small enough to ensure "good" properties of Π.
- Common choices for $\mathcal{A}$: CL(X) and C(X)
  CL(X) is the family of all closed (nonempty) subsets of X, whereas C(X) is the family of all compact (nonempty) subsets of X.

Standard constructions (2): miss-convergences

- Again, usual convergence of nets in a topological space X:

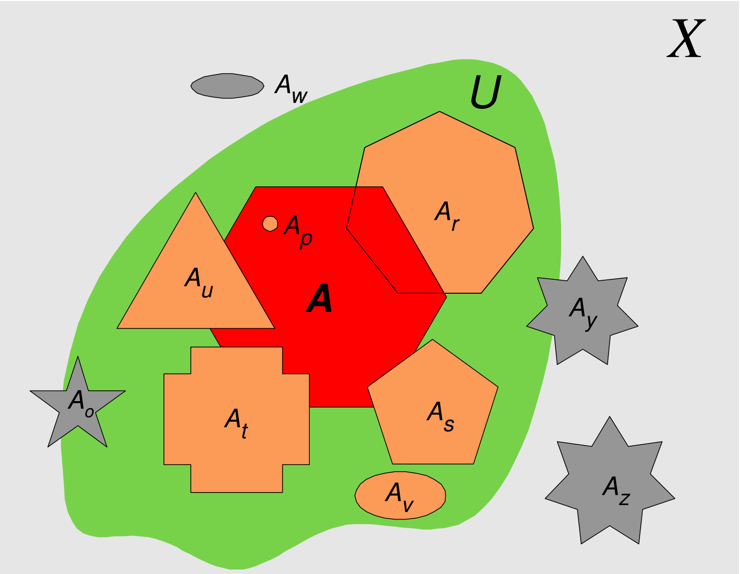
  $x_t \to x$ if for every open set U, $x \in U$ implies that $x_t \in U$ eventually.

- But $y \in U \Leftrightarrow \{y\} \subseteq U$, thus $x_t \to x$ if and only if for every open set U with $\{x\} \subseteq U$ we have $\{x_t\} \subseteq U$ eventually.

- Following this, we can define for a net (A_t) of subsets of X

  $A_t \to A$ if for every open set U, $A \subseteq U$ implies that $A_t \subseteq U$ eventually.

  This convergence is usually called the ***upper Vietoris*** convergence V^+.



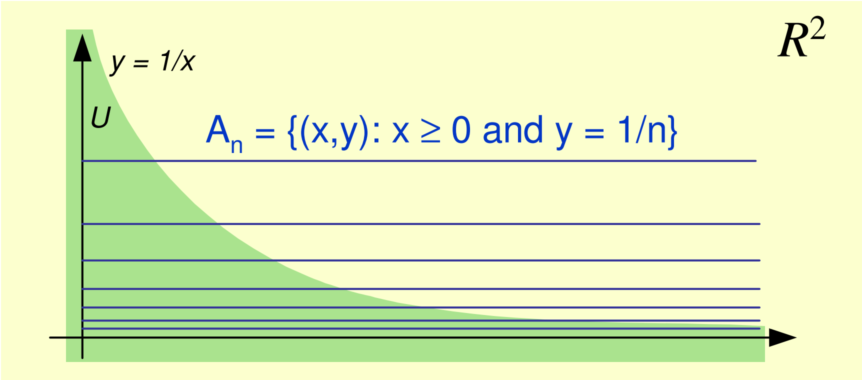
- Since $A \subseteq U \Leftrightarrow A \cap (X \setminus U) = \emptyset$,

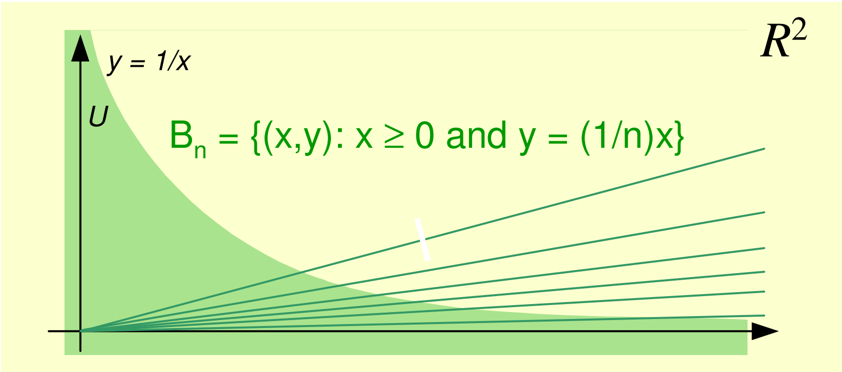
  V^+-convergence is an example of a so-called ***miss-convergence***:

  if A misses a closed set C then A_t misses C eventually.

Upper Vietoris convergence: basic properties (1)

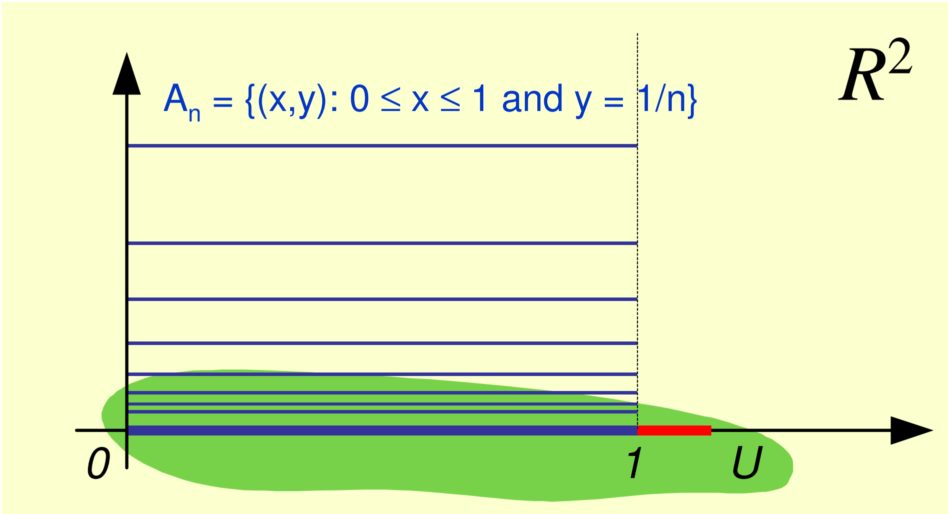
- V^+ is clearly admissible: $x_t \to x$ iff $\{x_t\} \to \{x\}$ with respect to V^+.
- The convergence V^+ is actually a topology: the family $\{\downarrow U: U \text{ is open}\}$, where $\downarrow U = \{A \subseteq X: A \subseteq U\}$, is a base of a topology on 2^X compatible with V^+.
- Is V^+ a “good” convergence?
  - V^+ is pretty strong: the sequences (A_n) and (B_n) do ***not*** converge to the semiline $\{(x,y) : x \geq 0 \text{ and } y = 0\}$ because $A_n \not\subset U$ and $B_n \not\subset U$ for every n





Upper Vietoris convergence: basic properties (2)

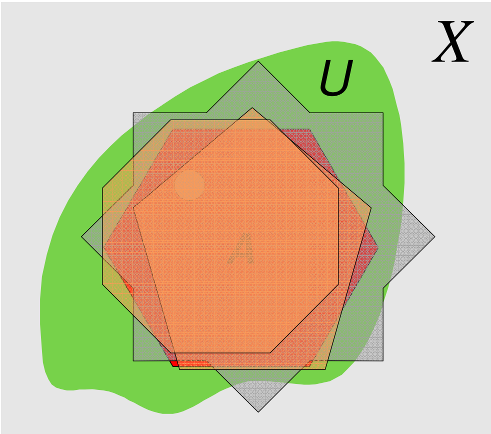
- It is clear that V^+ is the weakest set topology in which all sets of the form $\downarrow U$ are open, where $U \subseteq X$ is open.
- Notice that if $A_t \to A$ then $A_t \to B$ for ***any*** overset B of A (with respect to V^+).

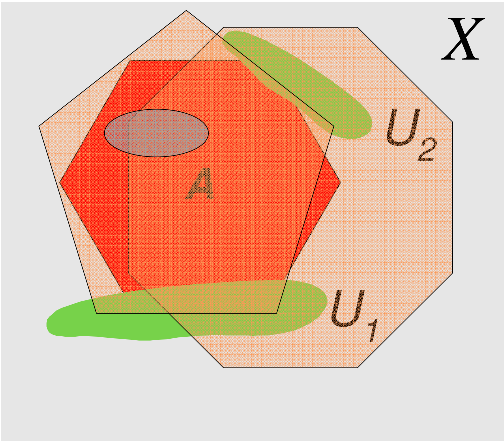


- Although V^+ is strong, it is still too coarse for good separation properties: V^+ is never T_1 (if card $X > 1$).
- However, if X is T_1 then V^+ is T_0
  - If $A \neq B$ then e.g. there is $x \in A \setminus B$. Since $\{x\}$ is closed, the set $\downarrow(\{x\}^c)$ is a V^+–neighborhood of B which does not contain A.

Vietoris convergence (1)

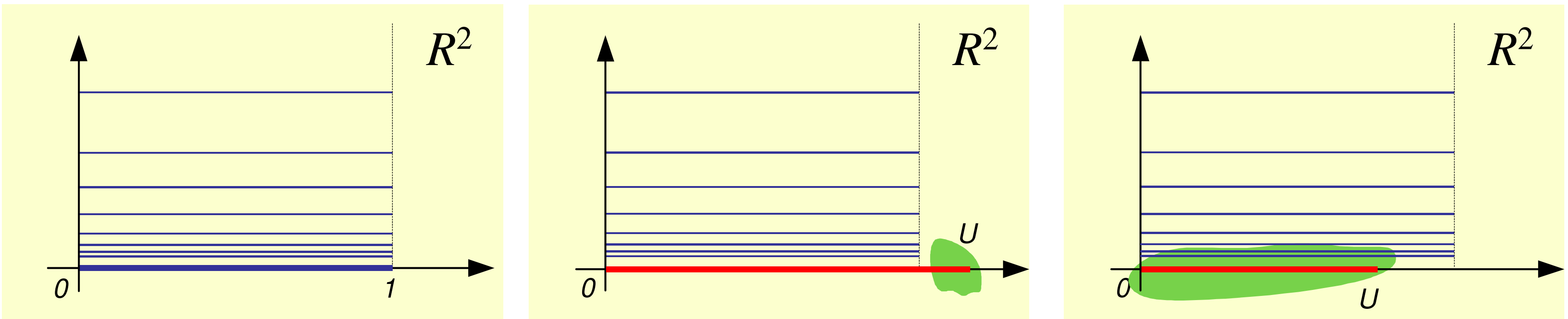
- The supremum convergence $V = V^{-} \vee V^{+}$ is called the ***Vietoris*** convergence.
- This convergence (which is a topology) was introduced by Leopold Vietoris (1891–2002) more than eighty years ago (in 1922) when he was looking for a convenient notion of manifold.
- The Vietoris topology is sometimes called ***finite topology*** (Michael [1951]).
- Although V is too strong in order to pass our “goodness” tests, it is more “balanced” as compared to its parts V^{-} and V^{+}.
- The intuitive idea behind the Vietoris topology is that, given an element A of 2^X, a basic V^{+}–neighborhood of A consists of sets B that are not much larger than A, and a basic V^{-}–neighborhood of A consists of sets C that are not much smaller than A.





Vietoris convergence (2)

- The convergences V^- and V^+ are not comparable



- The Vietoris convergence is not designed to distinguish between sets with the same closure.
- So it is usually considered at most on the family CL(X) of all (nonempty) subsets of X.
- For example, if X is regular then (CL(X), V) is Hausdorff.
- For more details on the Vietoris topology see E. Michael [1951].

Always trouble with the empty set

- Every net (A_t) of subsets of X converges to the empty set $\varnothing$ with respect to V^-.
- On the other hand, $\varnothing$ is an isolated point in $(2^X, V^+)$ and hence also in $(2^X, V)$.
- Thus usually the smaller family $P_0(X) = 2^X \setminus \{\varnothing\}$ (or some its subfamily) is considered.
- Another reason for it are multifunctions, i.e. set-valued mappings $y \to F(y) \neq \varnothing$.

What is “upper” and what is “lower” ?

- The designation of which convergence is “upper” and which is “lower” is arbitrary.
- This terminology was introduced by E. Michael (1951).
- There is no agreed general definition of “upper” or “lower” convergence.
- For purposes of this lecture we can consider the following definitions
  - A set convergence Π is called ***upper*** if $A \in \mathrm{Lim}^{\Pi} A_t$ implies that $\uparrow A \subseteq \mathrm{Lim}^{\Pi} A_t$
  - A set convergence Π is called ***lower*** if $A \in \mathrm{Lim}^{\Pi} A_t$ implies that $\downarrow A \subseteq \mathrm{Lim}^{\Pi} A_t$

Standard constructions (3): Limits of nets of subsets

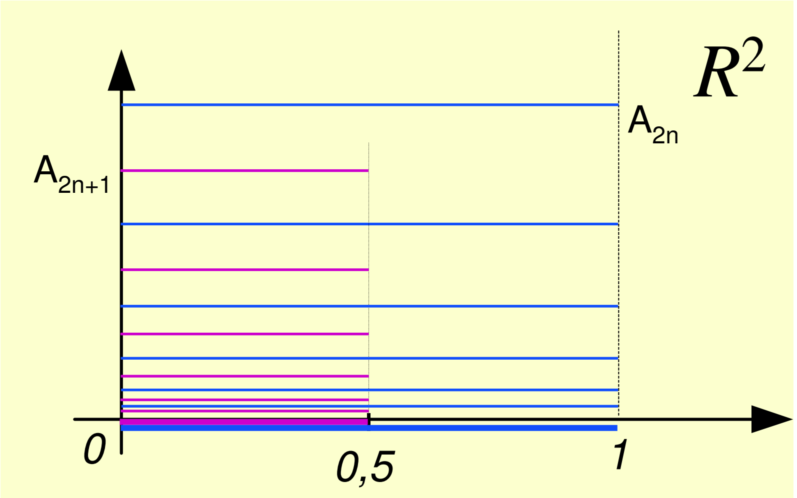
- For a net $(A_t)_{t\in T}$ of subsets of a topological space X we define two limit sets:
  - the lower limit

    $$\mathrm{Li}\, A_t = \{x\in X: \forall U\in N(x)\ \exists t \in T\ \forall s\geq t: A_s \cap U \neq \emptyset\}$$

  - and the upper limit

    $$\mathrm{Ls}\, A_t = \{x\in X: \forall U\in N(x)\ \forall t \in T\ \exists s\geq t: A_s \cap U \neq \emptyset\}$$

- Let us consider all nets $(x_t)_{t\in T}$ such that $x_t \in A_t$ for $t \in T$. The intuitive idea behind the above limits is that, the lower limit $\mathrm{Li}\, A_t$ is the set of all limits of the nets $(x_t)_{t\in T}$ whereas $\mathrm{Ls}\, A_t$ is the set of cluster points of such nets.
- Thus $\mathrm{Ls}\, A_t = \bigcap_t \overline{(\bigcup_{s\geq t} A_s)}$ and of course, $\mathrm{Li}\, A_t \subseteq \mathrm{Ls}\, A_t$
- In general the lower limit is strictly smaller than the upper one.



Limits of nets of subsets

- For a net $(A_t)_{t \in T}$ of subsets of a topological space X we say that $(A_t)_{t \in T}$ is ***K^--convergent*** to a subset A of X if $A \subseteq \text{Li } A_t$.
- The net $(A_t)_{t \in T}$ is called ***K^+-convergent*** to A if $\text{Ls } A_t \subseteq A$.
- And, the net $(A_t)_{t \in T}$ is called ***K-convergent*** to A if $\text{Ls } A_t \subseteq A \subseteq \text{Li } A_t$, i.e., $\text{Ls } A_t = A = \text{Li } A_t$
- It is clear that $K = K^- \vee K^+$.
- Of course, if $A_t \to A$ with respect to K then A is closed.
- The set convergence K is known as ***Painlevé-Kuratowski convergence*** (and usually called simply: Kuratowski convergence).
- K^- is also called the ***lower*** and K^+ the ***upper Kuratowski convergence*** – respectively.

Painlevé-Kuratowski convergence (1)

- The K-convergence has often been associated with Kuratowski but really has a much longer history starting with Painlevé (1902)
- It is easy to see that $K^{-} = V^{-}$. Consequently, K^{-} is admissible and passes our "goodness" tests (A) and (B)
- Since for a net (x_t) the upper limit Ls $\{x_t\}$ is equal to the set of all accumulation points of (x_t), K^{+} is not admissible in general (K^{+} is too coarse to make $\{x\} \rightarrow x$ continuous)
- The convergence K^{+} passes our "goodness" tests (A) and (B) because

$$\text{Ls } A_t = \text{Ls } B_t = \{(x,y): x \geq 0 \text{ and } y = 0\}$$

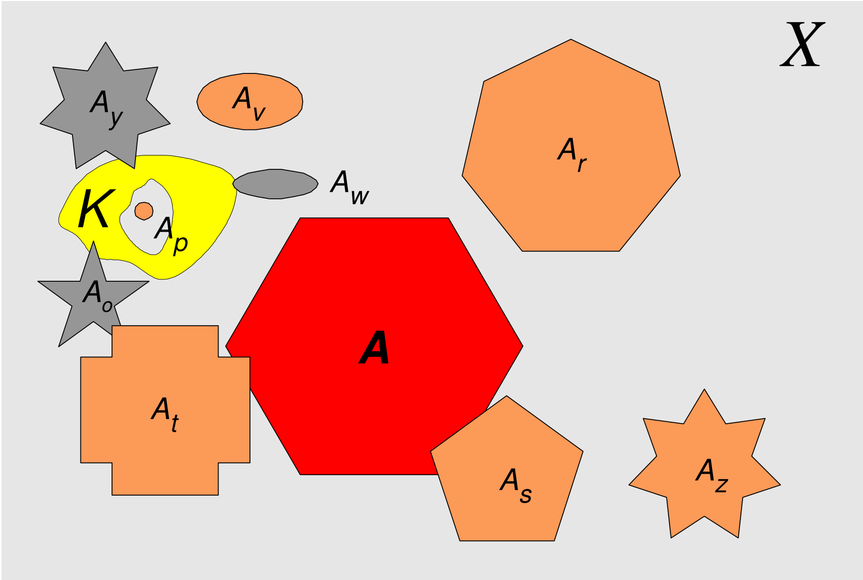
- In general K^{+} and V^{+} are not comparable. However, if X is regular and A is closed then $A_t \xrightarrow{V^+} A \Rightarrow A_t \xrightarrow{K^+} A$, i.e. $V^{+} \geq K^{+}$ on CL(X).
- Consequently, $V \geq K$ on CL(X) provided X is regular.

Painlevé-Kuratowski convergence (2)

- It is well known that neither the convergence K^+ nor K is topological in general
- Clearly, if $A_t \to A$ and $A_t \to B$ with respect to K then A = B
- Consequently, K is a Hausdorff convergence
- K is admissible if and only if X is Hausdorff
- For a net $(A_t)_{t\in T}$ let $\mathcal{F}$ be the filter on T generated by the sets of the form $\{s \in T: s \geq t\}$, $t \in T$.
- Let $\#\mathcal{F}$ denote the ***grill*** of $\mathcal{F}$, i.e. $\#\mathcal{F} = \{B \subseteq X: B \cap F \neq \emptyset \text{ for every } F \in \mathcal{F}\}$.
  Of course, $\mathcal{F} \subseteq \#\mathcal{F}$.
- For two filters $\mathcal{G}$ and $\mathcal{H}$ if $\mathcal{G} \subseteq \mathcal{H}$ then $\#\mathcal{H} \subseteq \#\mathcal{G}$. Moreover, if $\mathcal{U}$ is an ultrafilter then $\mathcal{U} = \#\mathcal{U}$.
- Observe that $\mathrm{Li}\, A_t = \bigcap_{H\in\#\mathcal{F}} (\overline{\bigcup_{s\in H} A_s})$ and $\mathrm{Ls}\, A_t = \bigcap_{F\in\mathcal{F}} (\overline{\bigcup_{s\in F} A_s})$.
- Now we can state the following remarkable property of the K convergence:
  it is always compact!

Another miss-convergence

- Recall that the V^+-convergence is an example of a so-called ***miss-convergence***:
  if A misses a closed set C then A_t misses C eventually.
- If we replace closed sets with compact sets we get the ***upper Fell*** convergence F^+ (called also ***co-compact*** convergence):
  if A misses a compact set K then A_t misses K eventually.
- F^+ is actually a topology, which is not admissible in general (F^+ is too coarse to make $\{x\} \to x$ continuous)
- Of course, F^+ passes our "goodness" tests (A) and (B)
- The supremum convergence $F = V^- \vee F^+$ is called ***Fell convergence*** (topology).

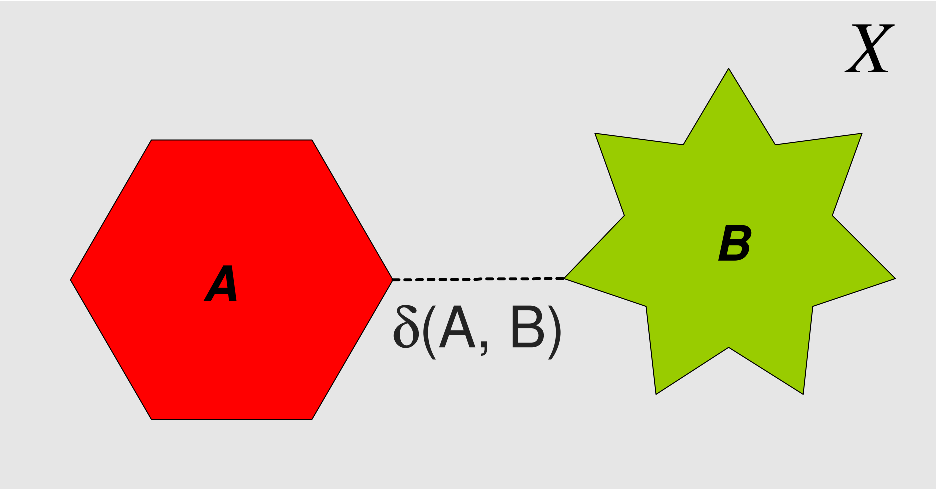


Fell convergence

- The convergence $F = V^{-} \vee F^{+}$ (elsewhere called the ***topology of closed convergence***) was introduced by J. Fell (1962).
- Of course, if X is Hausdorff then $V^{+} \geq F^{+}$ and consequently, $V \geq F$.
- The convergence F being weaker than V has proved to be the superior construct in terms of applications (particularly applications to optimization, convex analysis, mathematical economics, etc.).
- F^{+} turns out to be weaker than K^{+}, i.e. $K \geq F$. Consequently, F is also a compact (possibly non-Hausdorff) convergence!
- If X is locally compact (i.e. if every point of X has a neighborhood base consisted of compact sets) then F is Hausdorff, no matter how badly unseparated X may be.
- Since $K \geq F$ and K is a compact Hausdorff pseudotopology (which are minimal Hausdorff convergences), $F = K$ if X is locally compact (the converse is also true).
- F is admissible if and only if X is Hausdorff

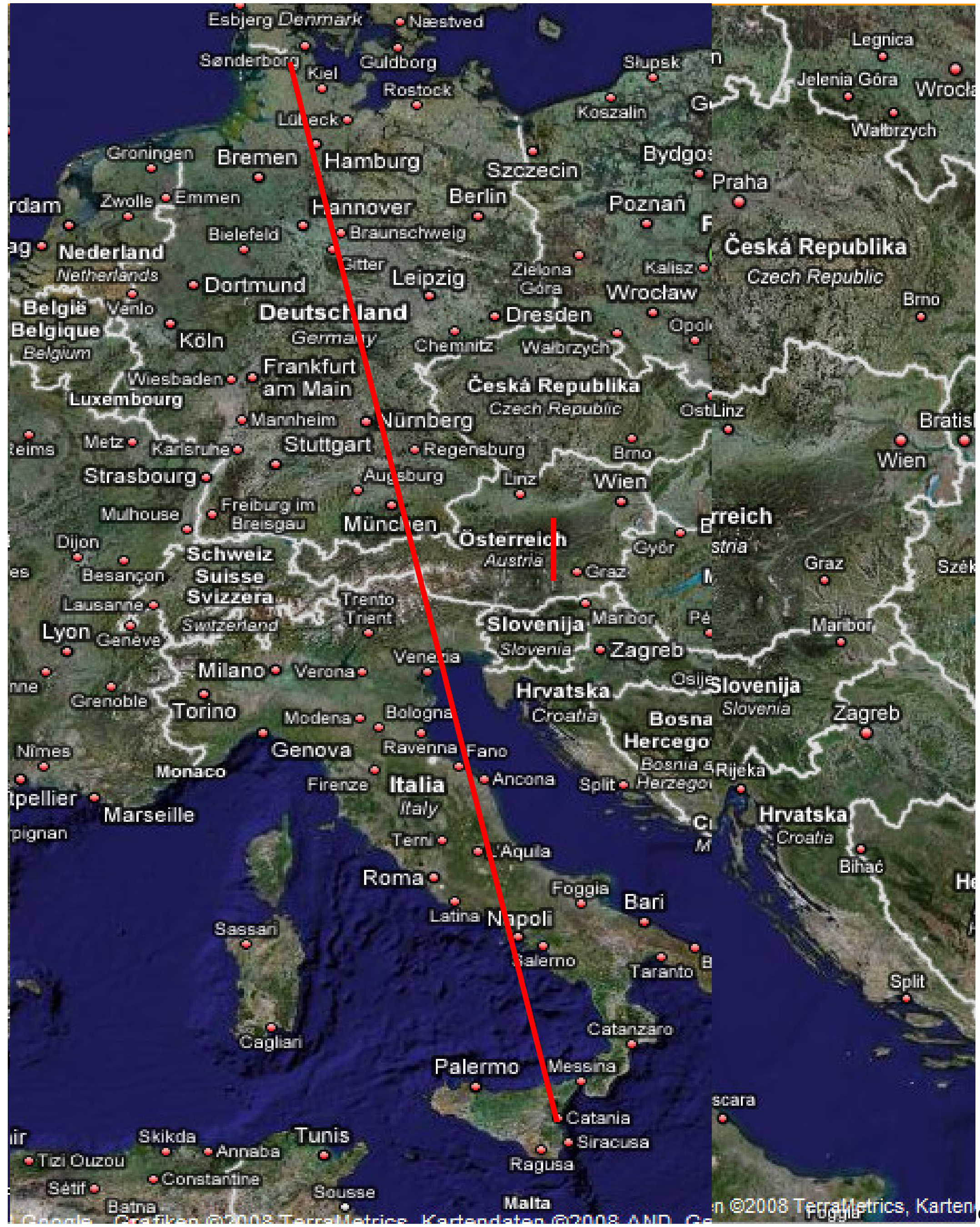
The metric case: How to define distance between sets?

- Now let (X, d) be a metric space. Questions:
  - Can we use the metric structure of X to construct a set convergence?
  - Can we define a metric on 2^X ?
- How to measure "distance" between sets?
- What about the gap function $(A, B) \to \delta(A, B) = \inf\{d(a, b): a \in A \text{ and } b \in B\}$?
- Of course, δ is not a metric but still one could try to define a convergence
  $A_t \to A$ iff $\delta(A_t, A) \to 0$
- However, this convergence is too coarse to be interesting.



Example

- What is the distance between Italy and Germany?
- The distance between Munich and Milan is < 400 km
- The distance between nearest points is < 100 km
- But what about a person traveling from Flensburg to Catania?



Standard constructions (4): Upper convergence of Hausdorff

- Now consider the usual convergence of nets in a metric space (X,d):

  $x_t \to x$ if for every $\varepsilon > 0$, $x_t \in B_\varepsilon(x)$ eventually.

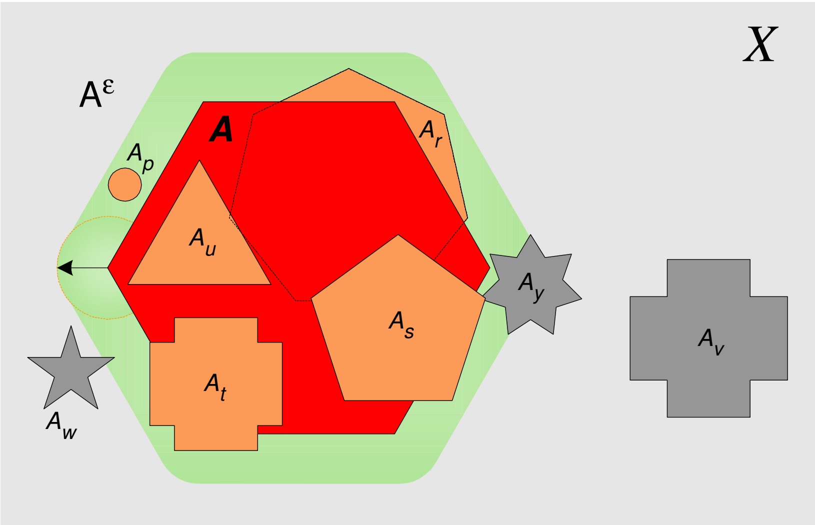
- For $A \subseteq X$ let A^ε be the ε-enlargement of the set A of radius ε. Then

  $x_t \to x$ if for every $\varepsilon > 0$, $\{x_t\} \subseteq B_\varepsilon(x) = \{x\}^\varepsilon$ eventually

- Now, for a net (A_t) of subsets of X we can now define

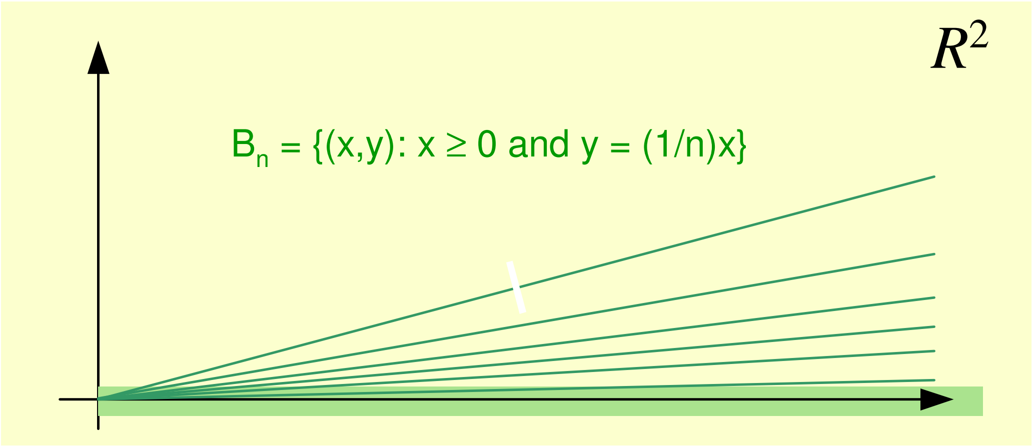
  $A_t \to A$ if for every $\varepsilon > 0$, $A_t \subseteq A^\varepsilon$ eventually,

  This is the so-called ***upper Hausdorff*** convergence H^+.



Upper Hausdorff convergence: basic properties

- H^+ is clearly admissible: $x_t \to x$ iff $\{x_t\} \to \{x\}$ with respect to H^+.
- The convergence H^+ is actually a topology: the family $\{\downarrow(A^\varepsilon): \varepsilon > 0\}$ is a local base of A for a topology on 2^X compatible with H^+.
- Is H^+ a "good" convergence?
  - H^+ is obviously ***coarser*** than V^+ : $V^+ \geq H^+$. Moreover $H^+ \geq K^+ \geq F^+$.
  - H^+ passes only our "goodness" test (A). The sequence (B_n), however, does ***not*** converge to the semiline $\{(x,y) : x \geq 0 \text{ and } y = 0\}$



Standard constructions (5): Lower convergence of Hausdorff

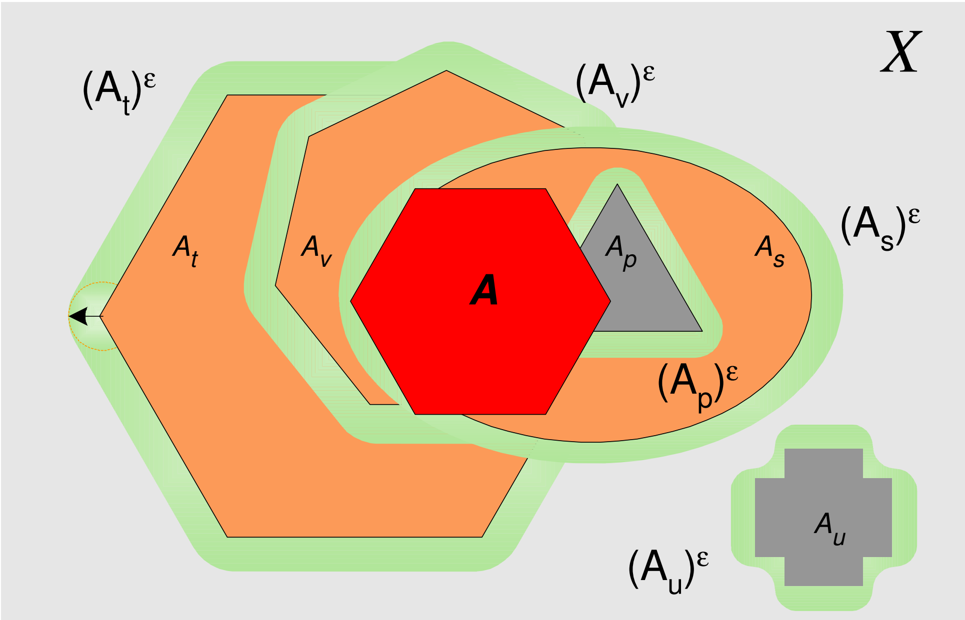
- Since $x_t \in \{x\}^\varepsilon$ if and only if $\{x\} \subseteq \{x_t\}^\varepsilon$ we can write

  $x_t \to x$ iff for every $\varepsilon > 0$, $\{x\} \subseteq \{x_t\}^\varepsilon$ eventually.

- For a net (A_t) of subsets of X we can thus define

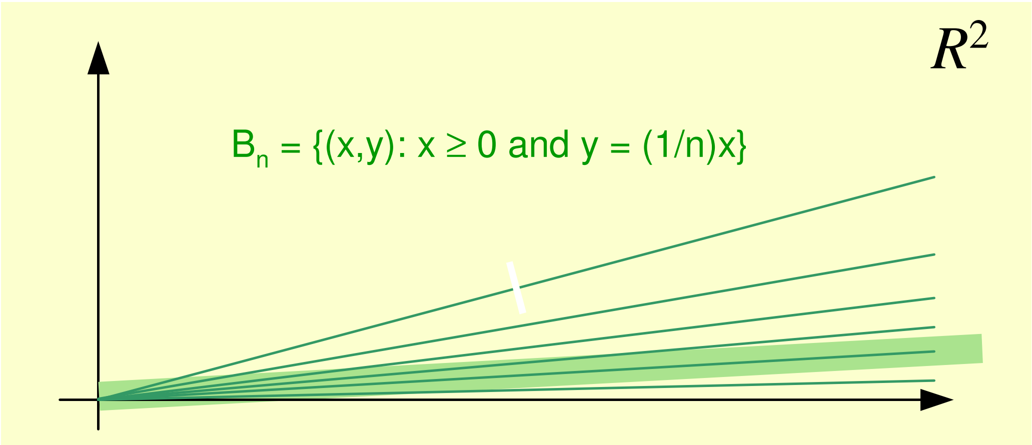
  $A_t \to A$ if for every $\varepsilon > 0$, $A \subseteq A_t^\varepsilon$ eventually,

  This is the so-called ***lower Hausdorff*** convergence H^-.



Lower Hausdorff convergence: basic properties

- H^- is clearly admissible: $x_t \to x$ iff $\{x_t\} \to \{x\}$ with respect to H^-.
- The convergence H^- is actually a topology: the family of all sets $\{B \subseteq X: A \subseteq B^{\varepsilon}\}$, $\varepsilon > 0$, is a local base of A for a topology on 2^X compatible with H^-.
- Is H^- a "good" convergence?
  - H^- is ***finer*** than V^-: $H^- \geq V^-$
  - H^- passes only our "goodness" test (A). The sequence (B_n), however, does ***not*** converge to the semiline $\{(x,y) : x \geq 0 \text{ and } y = 0\}$

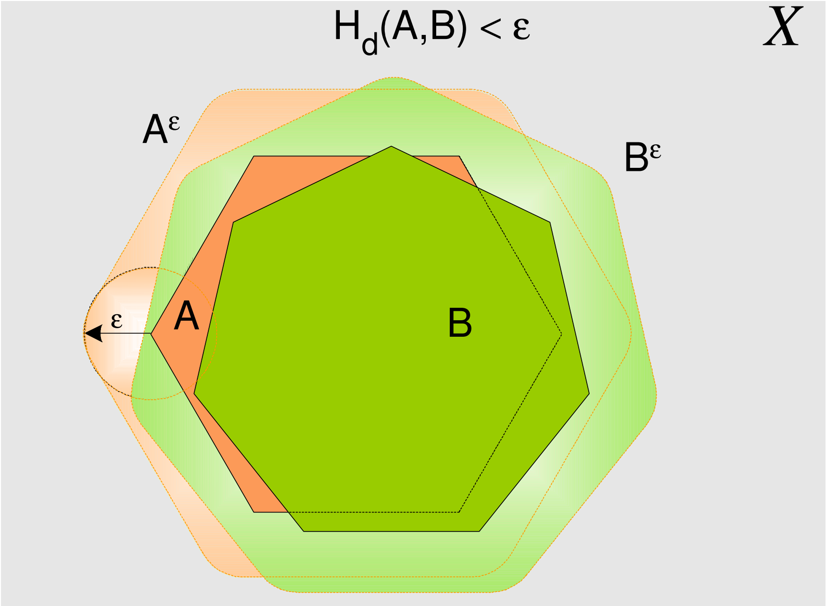


Standard constructions (6): Hausdorff distance

- The supremum convergence $H = H^{-} \vee H^{+}$ is called ***Hausdorff set convergence***
- H is generated by the so-called ***Hausdorff distance*** (Pompeiu 1905, Hausdorff 1912)

$$H_d(A,B) = \inf \{\varepsilon > 0: A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\} = \max \{\sup_{a\in A} \delta(a,B), \sup_{b\in B} \delta(b,A)\}.$$

- If $H_d(A,B) < \varepsilon$ for some $\varepsilon > 0$, we can say that "A *is not much larger than* B" and "B *is not much smaller than* A"



- In general, $V \neq H$ and $H \geq K \geq F$.
- H_d restricted to CL(X) is a metric (but can have infinite values)
- The convergence H (as well as H^{-} and H^{+}) can be easily formulated in the case of uniform spaces
- The convergence H is not a topological concept: two equivalent metrics on X need not lead to the same convergence.

Another look at the Hausdorff distance

- It is well known that

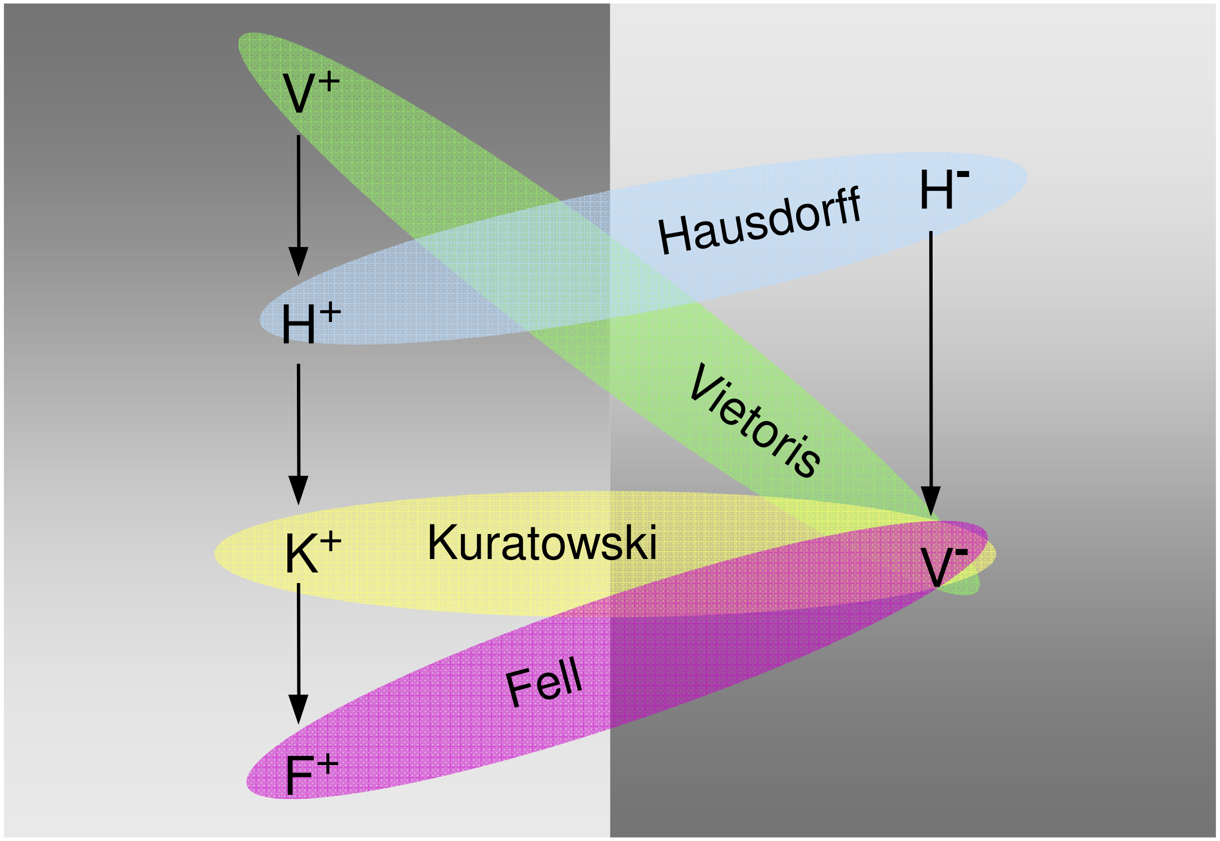
$$H_d(A,B) = \sup_{x\in X} |\delta(x,A) - \delta(x,B)|$$

- Consequently

$A_t \rightarrow A$ with respect to H_d if $\delta(x, A_t) \rightarrow \delta(x, A)$ uniformly on X

Set convergences: Summary

- Upper convergences



- Lower convergences

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My lecture next time (in the fall 2008?)

- How to improve the Hausdorff convergence?
- What are consonant and what dissonant spaces?
- What are bornologies good for?
- … and much more …

Questions?